LP -THEORY FOR VECTOR POTENTIALS AND SOBOLEV'S INEQUALITIES FOR VECTOR FIELDS. APPLICATION TO THE STOKES EQUATIONS WITH PRESSURE BOUNDARY CONDITIONS

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L^p-THEORY FOR VECTOR POTENTIALS AND SOBOLEV'S INEQUALITIES FOR VECTOR FIELDS. APPLICATION TO THE STOKES EQUATIONS WITH PRESSURE BOUNDARY CONDITIONS

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In a three dimensional bounded possibly multiply-connected domain, we give gradient and higher order estimates of vector fields *via* div and **curl** in L^p theory. Then, we prove the existence and uniqueness of vector potentials, associated with a divergence-free function and satisfying some boundary conditions. We also present some results concerning scalar potentials and weak vector potentials. Furthermore, we consider the stationary Stokes equations with nonstandard boundary conditions of the form $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}$ and $\pi = \pi_0$ on the boundary Γ. We prove the existence and uniqueness of weak, strong and very weak solutions. Our proofs are based on obtaining $Inf - Sup$ conditions that play a fundamental role. We give a variant of the Stokes system with these boundary conditions, in the case where the compatibility condition is not verified. Finally, we give two Helmholtz decompositions that consist of two kinds of boundary conditions such as $\boldsymbol{u} \cdot \boldsymbol{n}$ and $\boldsymbol{u} \times \boldsymbol{n}$ on Γ.

Keywords: Vector Potentials; boundary conditions; Stokes; Helmholtz decomposition; Inf-Sup condition; Sobolev inequality.

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1. Introduction

In many problems of fluids mechanics, the operators div and **curl** play an important role in the mathematical study of these problems. In particular, we need some inequalities to estimate the gradient of vector fields *via* div and **curl** as for instance:

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\big), \tag{1.1}$$

for all $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega)$, where Ω is a bounded open set of $\mathbb{R}^3$ with boundary Γ of class $\mathcal{C}^{1,1}$. However, in some physically problems, we need to consider vectors fields with either vanishing tangential components or vanishing normal components on the boundary. In this case, the inequality (1.1) is not true. Indeed, in the case where the first Betti number I or the second Betti number J do not vanish, the following kernels:

$$\boldsymbol{K}_N^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},$$
$$\boldsymbol{K}_T^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$$

have dimensions $I \geq 1$ and $J \geq 1$ respectively.

In this paper, we are interested in some inequalities of type (1.1), in the case where Ω has arbitrary Betti numbers and for vectors fields with vanishing tangential components or vanishing normal components on the boundary.

We assume that Ω is a connected subset of $\mathbb{R}^3$, such that:

(i) We do not assume that its boundary Γ is connected and we denote by Γ_i, $0 \leq i \leq I$, the connected components of Γ, Γ_0 being the boundary of the only unbounded connected component of $\mathbb{R}^3 \backslash \bar{\Omega}$. We also fix a smooth open set $\mathcal{O}$ with a connected boundary (a ball, for instance), such that $\bar{\Omega}$ is contained in $\mathcal{O}$, and we denote by Ω_i, $0 \leq i \leq I$, the connected component of $\mathcal{O}\backslash\bar{\Omega}$ with boundary Γ_i ($\Gamma_0 \cup \partial\mathcal{O}$ for $i = 0$).

(ii) We do not assume that Ω is simply-connected. Observe that each component Γ_i, $0 \leq i \leq I$, is an orientable manifold of dimension two and hence is homeomorphic to a torus with p_i holes (see Ref. 17 for these properties). We set $J = \sum_{i=0}^{I} p_i$ and we make the following assumption that permits to "cut "adequately Ω in order to reduce it to a simply-connected region.

In order to study the vector potentials, we have to describe with more precision the geometry of the domain. We first need the following definition.

Definition 1.1. A bounded domain in $\mathbb{R}^3$ is called pseudo-$\mathcal{C}^{1,1}$ if for any point $\boldsymbol{x}$ on the boundary there exist an integer $r(\boldsymbol{x})$ equal to 1 or 2 and a strictly positive real number ρ_0 such that for all real numbers ρ with $0 < \rho < \rho_0$, the intersection of Ω with the ball with centre $\boldsymbol{x}$ and radius ρ, has $r(\boldsymbol{x})$ connected components, each one being $\mathcal{C}^{1,1}$.

Hypothesis 1.1. There exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called "*cuts* ", contained in Ω, such that:

(i) each surface Σ_j is an open part of a smooth manifold $\mathcal{M}_j$,

(ii) the boundary of Σ_j is contained in $\partial\Omega$ for $1 \le j \le J$,

(iii) the intersection $\bar{\Sigma}_i \cap \bar{\Sigma}_j$ is empty for $i \ne j$,

(iv) the open set

$$\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$$

is pseudo-$\mathcal{C}^{1,1}$ simply-connected.

For $J = 1$ with $I = 3$, see for example Fig. 1.

The hypothesis (iv) is important but not common. In general, boundary values problems are solved in a Lipschitz domains. In section 3, we need to solve some elliptic problem (see Lemma 3.4) in Ω° which is not Lipschitz continuous. Moreover, the regularity $\mathcal{C}^{1,1}$ is necessary to obtain solutions in $\boldsymbol{W}^{1,p}(\Omega)$.

We need Sobolev spaces $\boldsymbol{W}^{s,p}(\Gamma_i)$ on the connected component Γ_i, for $0 \le i \le I$, $-2 < s < 2$ and for $1 < p < \infty$. We can also define Sobolev spaces on the cuts $\boldsymbol{W}^{s,p}(\Sigma_j)$ as restrictions to Σ_j of the distributions belonging to $\boldsymbol{W}^{s,p}(\mathcal{M}_j)$. We will note by $\boldsymbol{W}^{s,p}(\Sigma_j)'$ the dual space of $\boldsymbol{W}^{s,p}(\Sigma_j)$.

Let us introduce some notations. For any vector field $\boldsymbol{v}$ on Γ, we shall denote by v_n the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{n}$, while we shall denote by $\boldsymbol{v}_t$ the projection of $\boldsymbol{v}$ on the tangent hyperplane to Γ. In other words $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ and $\boldsymbol{v}_t = \boldsymbol{v} - v_n \boldsymbol{n}$. Let us now consider any point P on Γ and choose an open neighbourhood W of P in Γ small enough to allow the existence of two families of C^2 curves on W. The lengths s_1, s_2 along each family of curves, respectively, are a possible system of coordinates in W. We denote by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ the unit tangent vectors to each family of curves, respectively. With this notations, we have $\boldsymbol{v}_t = \sum_{k=1}^{2} v_k \boldsymbol{\tau}_k$, where $v_k = \boldsymbol{v} \cdot \boldsymbol{\tau}_k$.

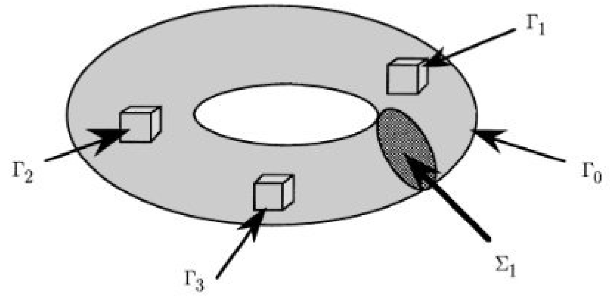

Fig. 1. .

We denote by $[\cdot]_j$ the jump of a function over Σ_j, *i.e.* the differences of the traces, for $1 \le j \le J$ and by $\langle \cdot, \cdot \rangle_{X,X'}$ the duality product between a space X and X'. We shall use bold characters for the vectors or the vector spaces and the non-bold characters for the scalars. The letter C denotes a constant that is not necessarily the same at its various occurrences. Finally, for any function q in $W^{1,p}(\Omega^\circ)$, $\mathbf{grad}\, q$ is the gradient of q in the sense of distributions in $\mathcal{D}'(\Omega^\circ)$. It belongs to $\boldsymbol{L}^p(\Omega^\circ)$ and therefore can be extended to $\boldsymbol{L}^p(\Omega)$. In order to distinguish this extension from the gradient of q in $\mathcal{D}'(\Omega)$, we denote it by $\widetilde{\mathbf{grad}}\, q$.

We shall show for every $1 < p < \infty$ the following first inequality concerning tangential vector fields:

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C(\|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}|), \tag{1.2}$$

and the second concernes the normal vector fields :

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C(\|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}|). \tag{1.3}$$

By means of the representation formula for $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and by introducing integral operators, Von Wahl,[28] obtained (1.2) and (1.3) without the flux through the cuts Σ_j for $1 \le j \le J$ and the components Γ_i for $1 \le i \le I$ on the right hand sides. So, he proved that such homogeneous estimates hold if and only if $I = 0$, *i.e.* Ω is simply connected in the case of $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ and if and only if $J = 0$, *i.e.* Ω has only one connected component of the boundary Γ in the case $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, respectively. In Ref. 9, the authors prove C^α-estimates of type (1.2) and (1.3) in a bounded smooth open set. Our estimates are then a generalization of Von Wahl's estimates which are a special case of ours. Our proofs are based on the Calderón Zygmund inequalities and the traces properties. Using the Peetre-Tartar Theorem, we deduce the following first Poincaré's inequality for every function $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ:

$$\|\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \le C(\|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{w}\|_{\mathrm{L}^p(\Omega)} + \sum_{\mathrm{i}=1}^{\mathrm{I}} |\langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_\mathrm{i}}|)$$

and a second for every function $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ:

$$\|\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \le C(\|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{w}\|_{\mathrm{L}^p(\Omega)} + \sum_{\mathrm{j}=1}^{\mathrm{J}} |\langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_\mathrm{j}}|)$$

Besides, we can deduce the following inequality by using the results in Section 3:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)}\big), \tag{1.4}$$

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}+\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big), \quad (1.5)$$

Moreover, we shall show the corresponding estimates for $\boldsymbol{v}$ in higher order Sobolev spaces $\boldsymbol{W}^{m,p}(\Omega)$ with $m \in \mathbb{N}^*$ via $\mathrm{div}\,\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u}$ when $\boldsymbol{v}\times\boldsymbol{n}$ or $\boldsymbol{v}\cdot\boldsymbol{n}$ does not vanish on Γ. These inequalities will be useful in order to prove regularity results of solution of Stokes problem and elliptic problems that we will solve. In addition, we will also consider the case of fractionary Sobolev spaces $\boldsymbol{W}^{s,p}(\Omega)$ with a real number s possibly not integer.

Next, we will give some generality results concerning vector, scalar potentials and weak vector potentials in $\boldsymbol{L}^p$ theory with $1 < p < \infty$, thus extending the results established by Amrouche, Bernardi, Dauge and Girault,[2] and by Amrouche, Ciarlet and Ciarlet, Jr,[3] in the hilbertian case (see also for exemple the results established by D. Mitrea, M. Mitrea and J. Pipher,[25]). In particular, we will prove existence of a first vector potential $\boldsymbol{\psi}$ associated with a divergence-free vector function $\boldsymbol{u}$ satisfying:

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \ \text{ on}\,\Gamma.$$

Using the classical Helmholtz decomposition and this tangential vector potential, we prove the following Inf-Sup condition:

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}_T(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}^{p}_T(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}^p_T(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}^{p'}_T(\Omega)}} > \beta_1, \quad (1.6)$$

where $\beta_1 > 0$ and the spaces $\boldsymbol{X}^p_T(\Omega)$, $\boldsymbol{V}^p_T(\Omega)$ are defined by

$$\begin{aligned}
\boldsymbol{X}^p_T(\Omega) &= \{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v}\in L^p(\Omega),\ \mathbf{curl}\,\boldsymbol{v}\in \boldsymbol{L}^p(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\},\\
\boldsymbol{V}^p_T(\Omega) &= \{\boldsymbol{v}\in \boldsymbol{X}^p_T(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\ \ 1\leq j\leq J\},
\end{aligned}$$

which plays a crucial role in the proof of the solvability of the following weak Neumann problem: for $\boldsymbol{v}\in \boldsymbol{L}^p(\Omega)$, find $\boldsymbol{u}\in \boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad (\mathbf{curl}\,\boldsymbol{u} - \boldsymbol{v})\times\boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases} \quad (1.7)$$

The problem (1.7) for the vector fields is in fact the equivalent of the Neumann problem for the scalar functions: for $\boldsymbol{f}\in \boldsymbol{L}^p(\Omega)$, find $\chi\in W^{1,p}(\Omega)$

$$-\Delta\,\chi = \mathrm{div}\,\boldsymbol{f} \ \text{ in } \Omega \quad \text{and} \quad (\nabla\,\chi - \boldsymbol{f})\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma. \quad (1.8)$$

As a consequence of the resolution of (1.7), we can prove the existence of a second vector potential satisfying:

$$\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \boldsymbol{\psi}\times\boldsymbol{n} = \mathbf{0} \ \text{ on}\,\Gamma$$

and then the following Inf-Sup condition:

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}_N(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}^{p}_N(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot \mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}^p_N(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}^{p'}_N(\Omega)}} > \beta_2, \tag{1.9}$$

where $\beta_2 > 0$ and the spaces $\boldsymbol{X}^p_N(\Omega)$, $\boldsymbol{V}^p_N(\Omega)$ are defined by

$$\boldsymbol{X}^p_N(\Omega) = \{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v}\in L^p(\Omega),\ \mathbf{curl}\,\boldsymbol{v}\in \boldsymbol{L}^p(\Omega) \text{ and } \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},$$
$$\boldsymbol{V}^p_N(\Omega) = \{\boldsymbol{v}\in \boldsymbol{X}^p_N(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0,\ 1\le i\le I\}.$$

As a consequence of the Inf-Sup condition (1.9), we can solve the following problem: for $\boldsymbol{v}\in \boldsymbol{L}^p(\Omega)$, find $\boldsymbol{u}\in \boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 & \text{in}\,\Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma, \end{cases} \tag{1.10}$$

which is then well-posed (see Section 5 for more general right hand sides in the dual space of $\boldsymbol{H}^{p'}_0(\mathbf{curl},\,\Omega)$). We can check this with another way. Indeed, the problem (1.10) is equivalent to:

$$\begin{cases} -\Delta\,\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{v} & \text{in }\Omega\\ \mathrm{div}\,\boldsymbol{u} = 0, \quad \text{and} \quad \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma, \end{cases}$$

but where we have replaced the condition $\mathrm{div}\,\boldsymbol{u} = 0$ in Ω by $\mathrm{div}\,\boldsymbol{u} = 0$ on Γ.

With the help of the results obtained on the vector potentials, we are able to describe Helmholtz decomposition of $\boldsymbol{L}^p$-vector fields on Ω in a more precise manner through solutions of the boundary value problems (1.7) and (1.10). So, we will prove the following $\boldsymbol{L}^p$-Helmholtz decomposition:

$$\boldsymbol{v} = \boldsymbol{z} + \nabla\,\chi + \mathbf{curl}\,\boldsymbol{u}, \tag{1.11}$$

where $\boldsymbol{z}\in \boldsymbol{K}^p_T(\Omega)$ is unique, $\chi\in W^{1,p}(\Omega)$ is unique up to an additive constant and $\boldsymbol{u}\in \boldsymbol{W}^{1,p}(\Omega)$ is the unique solution up to an additive element of $\boldsymbol{K}^p_N(\Omega)$ of the problem (1.10). A similar decomposition to (1.11) is recently shown by Kozono and Yanagisawa,[23] in the case of $\mathcal{C}^\infty$-boundary Γ by using the theory of Agmon-Douglis-Nirenberg. In the case $p = 2$, Buffa and Ciarlet. Jr in Ref. 11 and Ref. 12 obtained some Hodge decompositions on the boundary of Lipschitz polyhedra. See also Ref. 18 and Ref. 30. On the other hand, we will see that every $\boldsymbol{v}\in \boldsymbol{L}^p(\Omega)$ can be also decomposed as

$$\boldsymbol{v} = \boldsymbol{z} + \nabla\,\chi + \mathbf{curl}\,\boldsymbol{u}, \tag{1.12}$$

where $\chi \in W_0^{1,p}(\Omega)$, $\boldsymbol{z} \in \boldsymbol{K}_N^p(\Omega)$ are unique and $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ is the unique solution up to an additive element of $\boldsymbol{K}_T^p(\Omega)$ of the boundary value problem (1.7).

For $\boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$ with $s > 0$ and Ω of class $\mathcal{C}^\infty$, Bendali, Dominguez and Gallic,[6] obtained the decompositions (1.12) and (1.11).

As an application, we will consider the Stokes equations with non standard boundary conditions:

$$\begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = \chi & \text{in}\,\Omega, \\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\,\Gamma, \end{cases} \tag{1.13}$$

where $\boldsymbol{f}$, χ, $\boldsymbol{g}$, and π_0 are given functions. We denote by $\boldsymbol{n}$ the outward normal vector to Γ and the variables $\boldsymbol{u}$ and π usually represent velocity and pressure in fluid mechanics. In some physical situations, it is reasonable to prescribe the pressure on some part of the boundary as for instance in the case of blood vessels, pipelines. This boundary condition is naturally not sufficient to obtain a well-possed problem. In addition, we need to prescribe a second condition relating here to the tangential part of the velocity on the boundary. In the literature, many authors treat the case of mixed boundary conditions both numerical and theoretical as for instance (Ref. 14, Ref. 13, Ref. 7, Ref. 8, Ref. 20, Ref. 21, Ref. 24). Such problem come up in many practical applications *e.g.* fluids mechanic, electromagnetic fluids applications and decomposition of vector fields. However, for now, there are only theoretical papers on the solvability of such problem in Hilbert spaces. We propose in our work to develop a L^p theory to solve the problem (1.13). We prove existence and regularity of solutions for any $1 < p < \infty$. We also give a proof of the existence of a very weak solution when data are not regular enough, based on density arguments and functional framework adequate to define rigorously the trace of the vector functions which are living in subspaces of $\boldsymbol{L}^p(\Omega)$. Many authors consider a mixed method to solve (1.13), using a vector potential $\boldsymbol{\psi}$ satisfying $\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{u}$. Due to the non standard boundary conditions:

$$\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}, \quad \pi = \pi_0 \qquad \text{on }\Gamma,$$

the pressure is decoupled from the system. More precisely, we find that π is a solution of the problem:

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} + \Delta\,\chi \quad \text{in }\Omega \quad \text{and} \quad \pi = \pi_0 \ \text{ on }\Gamma.$$

Then, π can be found independently of $\boldsymbol{u}$. Observe that if $\operatorname{div}\boldsymbol{f} + \Delta\,\chi = 0$ in Ω and $\pi_0 = 0$ on Γ, the pressure π is zero, unlike the Stokes problem with Dirichlet boundary condition, where the pressure can not be constant.
With π known, we set $\boldsymbol{F} = \boldsymbol{f} - \nabla\,\pi$ and we obtain a system of equations involving only the velocity variable $\boldsymbol{u}$, that is:

$$\begin{cases} -\Delta\, \boldsymbol{u} = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = \chi & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on}\, \Gamma. \end{cases} \tag{1.14}$$

Here, we should remark that two different approachs to solve (1.14) are fully established. A first one by Schwarz,[26], where his method is based on the theory of pseudo-differential operator with the Lopatinski-Sapiro condition. The second one by Kozono and Yanagisawa,[23], where their method is based on the theory of Agmon-Douglis-Nirenberg. Kozono and Yanagisawa,[23] treat the case where Ω is a bounded domain of $\mathbb{R}^3$ with a $\mathcal{C}^\infty$-boundary Γ, $\boldsymbol{F} = \mathbf{curl}\, \boldsymbol{v}$ with $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{g} = \boldsymbol{0}$ and $\chi = 0$. Since the system (1.14) is not an elliptic boundary value problem in the sense of Agmon-Douglis-Nirenberg,[1] they rewrite (1.14) by replacing, in the case where $\chi = 0$ in Ω, the condition $\operatorname{div} \boldsymbol{u} = 0$ in Ω by $\operatorname{div} \boldsymbol{u} = 0$ on Γ. They verify that this modified problem is an elliptic boundary value problem in the sense of Agmon-Douglis-Nirenberg and they show that it fulfils the complementing condition in the sense of Agmon-Douglis-Nirenberg. But they consider that it is not possible to apply the theory of existence and regularity of solutions to the elliptic boundary problem to solve it, because, with a given $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$, they can only expect that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, so the value $\operatorname{div} \boldsymbol{u}$ on Γ cannot be well-defined. In fact, this is possible because due to Ref. 5, since $\operatorname{div} \boldsymbol{u} \in L^p(\Omega)$ and $\Delta \operatorname{div} \boldsymbol{u} \in W^{-1,p}(\Omega)$, the trace of $\operatorname{div} \boldsymbol{u}$ on Γ has a sense in $W^{-1/p,p}(\Gamma)$. Our proof of solvability of (1.14) is not based on the theory of Agmon-Douglis-Nirenberg but on a variational formulations and Inf-Sup conditions (1.6) and (1.9).

This paper is organized as follows. In Section 2, we will introduce some notations and we will state our main results. Section 3 is devoted to prove two kinds of Sobolev inequalities such as (1.2) and (1.3). Then, in Section 4, we will give some results concerning vector potentials depending on some boundary conditions on a given function $\boldsymbol{u}$. We will prove the existence and uniqueness of an associated vector potential also satisfying some gauge and boundary conditions. In the same section, we will treat the case of scalar potentials and weak vector potentials. In Section 5, we will solve problem (1.13). We will show the existence, uniqueness and regularity of the solution. We also study the existence of very weak solutions. Next, we will introduce a variant of the problem (1.13) which can be treated similarly but without assuming compatibility conditions. Finally, in Section 6, we will give the proof of the two Helmholtz decompositions (1.11) and (1.12).

2. Results

Before stating our results, we introduce some functions spaces. Let $\boldsymbol{L}^p(\Omega)$ denotes the usual vector-valued $\boldsymbol{L}^p-$space over Ω, $1 < p < \infty$. Let us define the spaces:

$$\boldsymbol{H}^{\,p}(\mathbf{curl}, \Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \right\},$$

with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathbf{curl},\Omega)} = \Big(\|\boldsymbol{v}\|^p{}_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\ \boldsymbol{v}\|^p{}_{\boldsymbol{L}^p(\Omega)}\Big)^{\frac{1}{p}},$$

$$\boldsymbol{H}^{\,p}(\mathrm{div},\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\ \boldsymbol{v} \in L^p(\Omega)\},$$

with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^p(\mathrm{div},\Omega)} = \Big(\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\ \boldsymbol{v}\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}}$$

and $\boldsymbol{X}^{\,p}(\Omega) = \boldsymbol{H}^{\,p}(\mathbf{curl},\Omega) \cap \boldsymbol{H}^{\,p}(\mathrm{div},\Omega)$, equipped with the graph norm. As in the case of Hilbert spaces, we can prove that $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{H}^{\,p}(\mathbf{curl},\ \Omega)$, $\boldsymbol{H}^{\,p}(\mathrm{div},\ \Omega)$ and $\boldsymbol{X}^{\,p}(\Omega)$.

We also define the subspaces:

$$\boldsymbol{H}^p_0(\mathbf{curl},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{\,p}(\mathbf{curl},\Omega);\ \ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},$$

$$\boldsymbol{H}^p_0(\mathrm{div},\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^{\,p}(\mathrm{div},\Omega);\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\},$$

$\boldsymbol{X}^{\,p}_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{\,p}(\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}$, $\boldsymbol{X}^{\,p}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{X}^{\,p}(\Omega); \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}$ and $\boldsymbol{X}^{\,p}_0(\Omega) = \boldsymbol{X}^{\,p}_N(\Omega) \cap \boldsymbol{X}^{\,p}_T(\Omega)$.

We have denoted by $\boldsymbol{v}\times\boldsymbol{n}$ (respectively $\boldsymbol{v}\cdot\boldsymbol{n}$) the tangential (respectively normal) boundary value of $\boldsymbol{v}$ defined in $\boldsymbol{W}^{\,-\frac{1}{p},p}(\Gamma)$ (respectively $W^{\,-\frac{1}{p},p}(\Gamma)$) as soon as $\boldsymbol{v}$ belongs to $\boldsymbol{H}^p(\mathbf{curl},\Omega)$ (respectively $\boldsymbol{H}^{\,p}(\mathrm{div},\Omega)$). More precisely, any function $\boldsymbol{v}$ in $\boldsymbol{H}^{\,p}(\mathbf{curl},\ \Omega)$ (respectively $\boldsymbol{H}^{\,p}(\mathrm{div},\Omega)$) has a tangential (respectively normal) trace $\boldsymbol{v}\times\boldsymbol{n}$ (respectively $\boldsymbol{v}\cdot\boldsymbol{n}$) in $\boldsymbol{W}^{\,-\frac{1}{p},p}(\Gamma)$ (respectively $W^{\,-\frac{1}{p},p}(\Gamma)$), defined by

$$\forall \boldsymbol{\varphi} \in \boldsymbol{W}^{\,1,p}(\Omega), \quad \langle \boldsymbol{v}\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}, \quad (2.1)$$

$$\forall \varphi \in W^{\,1,p}(\Omega), \quad \langle \boldsymbol{v}\cdot\boldsymbol{n}, \varphi\rangle_\Gamma = \int_\Omega \boldsymbol{v}\cdot\mathbf{grad}\,\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega (\mathrm{div}\,\boldsymbol{v})\varphi\,\mathrm{d}\boldsymbol{x}, \quad (2.2)$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality bracket between $\boldsymbol{W}^{\,-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{W}^{\,1-\frac{1}{p},p}(\Gamma)$ in (2.1) and between $W^{\,-\frac{1}{p},p}(\Gamma)$ and $W^{\,1-\frac{1}{p},p}(\Gamma)$ in (2.2).
We can prove as for the case $p = 2$ in Ref. 27 and Ref. 19, that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}^p_0(\mathbf{curl},\Omega)$ and in $\boldsymbol{H}^p_0(\mathrm{div},\Omega)$ for any $1 \le p < \infty$.

Our main results now reads as follows: first, we have the following gradient estimates of vector fields *via* div and **curl** and the quantities $\sum\limits_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|$ or $\sum\limits_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}|$ following the boundary condition that we consider. (Se Section 3)

Theorem 2.1.

(i) Any function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ *with* $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ *on* Γ *satisfies:*

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|\Big).$$

(ii) Any function $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ *with* $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ *on* Γ *satisfies:*

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}|\Big).$$

The following theorem gives two fundamental Inf-Sup conditions. (See Subsection 4.3 and Subsection 5.1)

Theorem 2.2. *The following Inf-Sup conditions hold :*

(i) There exist a constant $\beta_1 > 0$ *such that*

$$\inf_{\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)} \sup_{\boldsymbol{\xi}\in \boldsymbol{V}_N^{p}(\Omega)} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_N^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)}} > \beta_1,$$

(ii) There exist a constant $\beta_2 > 0$ *such that*

$$\inf_{\boldsymbol{\varphi}\in \boldsymbol{V}_T^{p'}(\Omega)} \sup_{\boldsymbol{\xi}\in \boldsymbol{V}_T^{p}(\Omega)} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_T^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}} > \beta_2.$$

The next result concerns the existence and uniqueness of the weak, strong and very weak solution of the problem $(\mathcal{S}_N)$. (See Section 5)

Theorem 2.3. **(Weak, Strong and Very weak solutions for** $(\mathcal{S}_\mathrm{N})$**)**

(i) Let $\boldsymbol{f}$, $\boldsymbol{g}$, π_0 *be such that*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))', \;\; \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \;\; \pi_0 \in W^{\,1-1/p,p}(\Gamma)$$

satisfying the compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)} - \int_\Gamma \pi_0\, \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{\sigma} = 0. \quad (2.3)$$

Then, the Stokes problem $(\mathcal{S}_N)$ *has a unique solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{\,1,p}(\Omega)$ *satisfying the estimate:*

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\,\pi\,\|_{W^{\,1,p}(\Omega)} \leq C\Big(\|\,\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))'} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \\ + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\Big).$$

(ii) Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, $\pi_0 \in W^{\,1-1/p,p}(\Gamma)$, *then the solution* $(\boldsymbol{u},\, \pi)$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega) \times W^{\,1,p}(\Omega)$ *and satisfies the estimate:*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{2,p}(\Omega)} + \| \pi \|_{W^{1,p}(\Omega)} \leq C\Big(\| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \| \boldsymbol{g} \times \boldsymbol{n} \|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)} + \| \pi_0 \|_{W^{1-1/p,p}(\Gamma)} \Big).$$

(iii) Let $\boldsymbol{f}$, $\boldsymbol{g}$, *and* π_0 *with*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]', \ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \ \pi_0 \in W^{-1/p,p}(\Gamma),$$

and satisfying the compatibility condition (2.3)*. Then, the Stokes problem* $(\mathcal{S}_N)$ *has exactly one solution* $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ *and* $\pi \in L^p(\Omega)$*. Moreover, there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{L}^p(\Omega)} + \| \pi \|_{L^p(\Omega)} \leq C\Big(\| \boldsymbol{f} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega)]'} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} + \| \pi_0 \|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \Big).$$

When the compatibilty condition (5.20) is not satisfied, we are reduced, as in Ref. 14, to solve a variant of the Stokes problem (see Subsection 5.4)

3. $\boldsymbol{L}^{\mathbf{p}}$-Sobolev Inequalities for Vector Fields

The aim of this section is to prove continuous imbeddings of both spaces $\boldsymbol{X}_T^p(\Omega)$ and $\boldsymbol{X}_N^p(\Omega)$ in $\boldsymbol{W}^{1,p}(\Omega)$. In a first step, we introduce an integral operator that allows to estimate $\nabla \boldsymbol{v}$ by $\mathbf{curl}\, \boldsymbol{v}$, $\operatorname{div} \boldsymbol{v}$ and the flux of $\boldsymbol{v}$ past the boundary Γ_i for $1 \leq i \leq I$ provided that $\boldsymbol{v} \times \boldsymbol{n} = \mathbf{0}$ on Γ. In a second step, we introduce another integral operator to estimate $\nabla \boldsymbol{v}$ by $\mathbf{curl}\, \boldsymbol{v}$, $\operatorname{div} \boldsymbol{v}$ and the flux of $\boldsymbol{v}$ past the cuts Σ_j, $1 \leq j \leq J$ provided that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ.

3.1. *Estimates with tangential boundary conditions*

We introduce the linear integral operator

$$T\,\lambda(\boldsymbol{x}) = -\frac{1}{2\pi} \int_\Gamma \lambda(\boldsymbol{\xi})\, \frac{\partial}{\partial \boldsymbol{n}} |\boldsymbol{x} - \boldsymbol{\xi}|^{-1}\, \mathrm{d}\sigma_{\boldsymbol{\xi}}.$$

The next lemma gives some properties of this operator.

Lemma 3.1. *We have the following properties:*

(i) The operator T *is compact from* $L^p(\Gamma)$ *into* $L^p(\Gamma)$*.*

(ii) The space $\operatorname{Im}(Id+T)$ *is a closed subspace of* $L^p(\Gamma)$ *and* $\operatorname{Ker}(Id+T)$ *is of finite dimension. It is spanned by the traces of the functions* $\mathbf{grad}\, q_i^N \cdot \boldsymbol{n}|_\Gamma$, $1 \leq i \leq I$, *where each* q_i^N *is the unique solution in* $W^{2,p}(\Omega)$ *of the problem*

$$\begin{cases} -\Delta q_i^N = 0 \qquad \text{in } \Omega, \\ q_i^N|_{\Gamma_0} = 0 \quad \text{and} \quad q_i|_{\Gamma_k} = \text{constant}, \ \ 1 \leq k \leq I, \\ \big\langle \partial_n\, q_i^N, \, 1 \big\rangle_{\Gamma_k} = \delta_{i\,k}, \ 1 \leq k \leq I, \quad \text{and} \quad \big\langle \partial_n\, q_i^N, \, 1 \big\rangle_{\Gamma_0} = -1, \end{cases} \tag{3.1}$$

(see Subsection 4.3).

(iii) For any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ *we have:*

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{L^p(\Gamma)} \leq C\big(\|(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n})\|_{L^p(\Gamma)} + \sum_{i=1}^{I}|\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}|\big). \tag{3.2}$$

Proof.

(i) According to Von Wahl,[28], we have $T \in \mathcal{L}(L^p(\Gamma), W^{1,p}(\Gamma))$. Since the embedding of the space $W^{1,p}(\Gamma)$ in $L^p(\Gamma)$ is compact, we obtain that T is compact from $L^p(\Gamma)$ into $L^p(\Gamma)$.

(ii) By virtue of the first point and the Fredholm alternative, we have that the space $\mathrm{Ker}(Id+T)$ is of finite dimension and $\mathrm{Im}(Id+T)$ is a closed subspace of $L^p(\Gamma)$. Since the functions $\mathbf{grad}\, q_i^N$, $1 \leq i \leq I$ are linearly independent, it is readily checked that $\mathbf{grad}\, q_i^N \cdot \boldsymbol{n}|_\Gamma$ are also linearly independent for $1 \leq i \leq I$ (for the properties of q_i^N, see Section 4).
Now, let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. Then the quantity $\boldsymbol{v}\cdot\boldsymbol{n}$ satisfies on Γ the following representation (see Ref. 28):

$$\begin{aligned}(Id+T)(\boldsymbol{v}\cdot\boldsymbol{n}) = &-\frac{1}{2\pi}\left(\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n}\\ &-\frac{1}{2\pi}\left(\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\cdot\boldsymbol{n}\\ &+\frac{1}{2\pi}\left(\mathbf{curl}\int_\Gamma \frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}(\boldsymbol{v}\times\boldsymbol{n})(\boldsymbol{\xi})\,\mathrm{d}\sigma_{\boldsymbol{\xi}}\right)\cdot\boldsymbol{n}.\end{aligned} \tag{3.3}$$

As $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{W}^{1,p}(\Omega)$, this relation is still valid for $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$. Since $\mathbf{grad}\, q_i^N$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and $\mathbf{grad}\, q_i^N \times \boldsymbol{n} = \mathbf{0}$ on Γ, due to (3.3), for each $1 \leq i \leq I$ the function $\mathbf{grad}\, q_i^N \cdot \boldsymbol{n}$ belongs to $\mathrm{Ker}(Id+T)$. Since the dimension of $\mathrm{Ker}(Id+T)$ is equal to I (see Ref. 28), the set $\{\mathbf{grad}\, q_i^N \cdot \boldsymbol{n}|_\Gamma,\ 1 \leq i \leq I\}$ is a basis of $\mathrm{Ker}(Id+T)$.

(iii) The operator $Id+T$ is linear, continuous and surjective from $L^p(\Gamma)$ onto $\mathrm{Im}(Id+T)$. Since $\mathrm{Ker}(Id+T)$ is of finite dimension, through the theorem of open application we deduce the existence of a constant $C > 0$ such that (3.2) holds. $\square$

The result of the next theorem is a generalization of the one in Ref. 28 to the case $I \geq 1$. So, we expect that for an estimate of $\nabla\,\boldsymbol{u}$ in addition to $\mathrm{div}\,\boldsymbol{v}$ and $\mathbf{curl}\,\boldsymbol{u}$ the quantity $\sum_{i=1}^{I}\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}$ if $\boldsymbol{v}\times\boldsymbol{n}$ vanish on Γ.

Theorem 3.1. *Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ be such that $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ. Then the following estimate holds*

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}|\big), \tag{3.4}$$

where the constant C depends only on p and Ω.

Proof. We use the same arguments as in Ref. 28 and we proceed in two steps. First, we prove that for any function $\boldsymbol{v}$ of $\boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ we have:

$$\|\boldsymbol{v} \cdot \boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq C\big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}|\big). \tag{3.5}$$

Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{v} \times \boldsymbol{n} = 0$ on Γ. By using the trace inequality and next the Caldéron-Zygmund inequality in the integral representation (3.3) we obtain

$$\|(Id + T)(\boldsymbol{v} \cdot \boldsymbol{n})\|_{L^p(\Gamma)} \leq C\left(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\right).$$

Thus, it follows directly by using (3.2) that

$$\|\boldsymbol{v} \cdot \boldsymbol{n}\|_{L^p(\Gamma)} \leq C\big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}|\big). \tag{3.6}$$

Moreover, from the equality (3.3), since $T \in \mathcal{L}\big(L^p(\Gamma),\, W^{1,p}(\Gamma)\big)$ and using the trace inequality, we obtain

$$\begin{aligned}\|\boldsymbol{v} \cdot \boldsymbol{n}\|_{W^{1-\frac{1}{p},p}(\Gamma)} \leq C\Big(&\|\boldsymbol{v} \cdot \boldsymbol{n}\|_{L^p(\Gamma)} + \Big\|\mathbf{grad} \int_\Omega \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, \mathrm{div}_{\boldsymbol{y}}\, \boldsymbol{v}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)} \\ &+ \Big\|\, \mathbf{curl} \int_\Omega \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\, \boldsymbol{v}(\boldsymbol{y})\, \mathbf{d}\boldsymbol{y}\Big\|_{\boldsymbol{W}^{1,p}(\Omega)}\Big).\end{aligned}$$

We use again the Calderón-Zygmund inequalities and (3.6) to obtain (3.5), which completes the proof of the first step.

Secondly, as $\boldsymbol{v}$ belongs to $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, due to the trace theorem, there exists a $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ such that

$$\boldsymbol{v} = \boldsymbol{u} \quad \text{on}\, \Gamma \qquad \text{and} \quad \|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}.$$

Since $\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0}$ on Γ, we have $\boldsymbol{v}|_\Gamma = (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$. Then, by using (3.5) we have

$$\begin{aligned}\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C\|\boldsymbol{v} \cdot \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} \\ &\leq C\Big(\|\mathrm{div}\, \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i}|\Big).\end{aligned} \tag{3.7}$$

We know that, for any function $\boldsymbol{w}$ of $\boldsymbol{W}_0^{1,p}(\Omega)$, we have the following integral representation

$$\boldsymbol{w} = -\mathbf{grad}\,\frac{1}{4\pi}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{w}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \mathbf{curl}\,\frac{1}{4\pi}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{w}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Using the Calderón-Zygmund inequalities, we have

$$\|\nabla\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\mathrm{div}\,\boldsymbol{w}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}\big). \tag{3.8}$$

Applying (3.8) to $\boldsymbol{v}-\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\Omega)$, we obtain

$$\|\nabla\,(\boldsymbol{v}-\boldsymbol{u})\|_{\boldsymbol{L}^p(\Omega)} \leq C\Big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{u}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}\Big).$$

Finally, the assertion (3.4) follows directly by using (3.7). □

Remark 3.1. We recall that if $p = 2$, any function $\boldsymbol{v}$ of $\boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_N^2(\Omega)$ satisfies (see Lemma 2.11 of Ref. 2)

$$\|\nabla\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\Omega)} - \int_\Gamma (\mathrm{Tr}\,\mathcal{B})(\boldsymbol{v}\cdot\boldsymbol{n})^2\,\mathrm{d}\boldsymbol{\tau},$$

where $\mathcal{B}$ is the curvature tensor of the boundary and $\mathrm{Tr}\,\mathcal{B}$ denote the trace of $\mathcal{B}$. In the case $p \neq 2$, we have the inequality (3.4) which allows us to estimate $\nabla\,\boldsymbol{v}$ by $\mathbf{curl}\,\mathbf{v}$, $\mathrm{div}\,\boldsymbol{v}$ and $\sum_{i=1}^{I} |\langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|$. Note that

$$|\int_\Gamma (\mathrm{Tr}\,\mathcal{B})(\boldsymbol{v}\cdot\boldsymbol{n})^2\,\mathrm{d}\boldsymbol{\tau}| \leq C\int_\Gamma |\boldsymbol{v}|^2 \leq \frac{1}{2}\|\nabla\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + C\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}.$$

We can then deduce the following inequality:

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^2(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\big). \tag{3.9}$$

Corollary 3.1. *Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ be such that $\boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ. Then, we have the following estimate:*

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\big). \tag{3.10}$$

Proof. Let $i \in [\,1,\, I\,]$ fixed. For any $\mu \in W^{1-1/p',p'}(\Gamma_i)$, we can find $\varphi \in W^{1,p'}(\Omega)$ such that

$$\varphi = \mu \ \text{ on } \Gamma_i \quad \text{and} \quad \varphi = 0 \ \text{ on } \Gamma_k \ \text{ for any } k \neq i$$

and satisfies the estimate

$$\|\varphi\|_{W^{1,p'}(\Omega)} \leq C\|\mu\|_{W^{1-1/p',p'}(\Gamma_i)}.$$

Moreover, if $\boldsymbol{v} \in \boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$, then

$$\langle \boldsymbol{v}\cdot\boldsymbol{n},\,\mu\rangle_{\Gamma_i} = \int_\Omega \boldsymbol{v}\cdot\nabla\,\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \varphi\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$

and taking $\mu = 1$, we obtain

$$|\langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}| \le C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)}\big).$$

We then deduce from (3.4) the estimate (3.10). $\square$

Now, we give the following density result. This result is proven in Ref. 2 for $p = 2$ and we give here a generalization in a different way for any $1 < p < \infty$.

Lemma 3.2. *The space* $\boldsymbol{W}^{1,p}(\Omega)\cap\boldsymbol{X}^p_N(\Omega)$ *is dense in the space* $\boldsymbol{X}^p_N(\Omega)$.

Proof. Let $\boldsymbol{\ell}$ belongs to $(\boldsymbol{X}^p_N(\Omega))'$, the dual space of $\boldsymbol{X}^p_N(\Omega)$. We know that there exist $\boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{g}\in\boldsymbol{L}^{p'}(\Omega)$ and $h\in L^{p'}(\Omega)$ such that

$$\langle \boldsymbol{\ell},\boldsymbol{v}\rangle = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega h\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{g}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}, \qquad \forall\boldsymbol{v}\in\boldsymbol{X}^p_N(\Omega). \tag{3.11}$$

We suppose that

$$\langle \boldsymbol{\ell},\,\boldsymbol{v}\rangle = 0 \quad \forall\boldsymbol{v}\in\,\boldsymbol{W}^{\,1,p}(\Omega)\cap\boldsymbol{X}^{\,p}_N(\Omega). \tag{3.12}$$

So, we have in the sense of distributions in Ω

$$\boldsymbol{f} - \nabla\,h + \mathbf{curl}\,\boldsymbol{g} = \boldsymbol{0}. \tag{3.13}$$

Therefore, due to (3.12) and (3.11), we have for any $\chi\in W^{\,2,p}(\Omega)\cap W^{\,1,p}_0(\Omega)$

$$\int_\Omega \boldsymbol{f}\cdot\nabla\chi\,\mathrm{d}\boldsymbol{x} + \int_\Omega h\,\Delta\chi\,\mathrm{d}\boldsymbol{x} = 0. \tag{3.14}$$

Note that $\mathrm{div}\,\boldsymbol{f} = \Delta h \in W^{\,1,p'}(\Omega)$. Because $h\in L^{p'}(\Omega)$, we know that $h_{|\Gamma}\in W^{\,-1/p',p'}(\Gamma)$ and we have (see [5]) for any $\chi\in W^{\,2,p}(\Omega)\cap W^{\,1,p}_0(\Omega)$

$$\int_\Omega h\,\Delta\,\chi\,\mathrm{d}\boldsymbol{x} - \langle\mathrm{div}\,\boldsymbol{f},\,\chi\rangle_{W^{\,-1,p'}(\Omega)\times W^{\,1,p}_0(\Omega)} = \langle\,h,\,\frac{\partial\,\chi}{\partial\boldsymbol{n}}\,\rangle_\Gamma.$$

As

$$\int_\Omega \boldsymbol{f}\cdot\nabla\chi\,\mathrm{d}\boldsymbol{x} = -\langle\mathrm{div}\,\boldsymbol{f},\,\chi\rangle_{W^{\,-1,p'}(\Omega)\times W^{\,1,p}_0(\Omega)},$$

it follows from (3.14) that

$$\langle\,h,\,\frac{\partial\,\chi}{\partial\boldsymbol{n}}\,\rangle_\Gamma = 0, \qquad \forall\chi\in W^{\,2,p}(\Omega)\cap W^{\,1,p}_0(\Omega). \tag{3.15}$$

Now, let μ be any element of $W^{\,1-\frac{1}{p},p}(\Gamma)$. Then, there exists an element χ of $W^{\,2,p}(\Omega)\cap W^{\,1,p}_0(\Omega)$ such that $\frac{\partial\chi}{\partial\boldsymbol{n}} = \mu$ on Γ. Hence, (3.15) implies that

$$\langle\,h,\,\mu\,\rangle_{W^{\,-\frac{1}{p'},p'}(\Gamma)\times W^{\,1-\frac{1}{p},p}(\Gamma)} = 0,$$

and $h = 0$ in $W^{-\frac{1}{p'},p'}(\Gamma)$. Because Δh belongs to $W^{-1,p'}(\Omega)$ and $h \in L^{p'}(\Omega)$, then $h \in W_0^{1,p'}(\Omega)$. As a consequence, due to (3.13), $\mathbf{curl}\, \boldsymbol{g}$ belongs to $\boldsymbol{L}^{p'}(\Omega)$. Finally, let $\boldsymbol{v}$ in $\boldsymbol{X}_N^p(\Omega)$. From (3.13) and since $h \in W_0^{1,p}(\Omega)$, we can write

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega h \operatorname{div} \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathbf{curl}\, \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = 0, \qquad \forall \boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega). \tag{3.16}$$

As $\boldsymbol{g} \in \boldsymbol{H}^{p'}(\mathbf{curl}, \Omega)$, we have also

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0^p(\mathbf{curl}, \Omega), \qquad \int_\Omega \mathbf{curl}\, \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{g} \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}.$$

Then it follows from the last equality and (3.16) that $\boldsymbol{\ell}$ vanishes on $\boldsymbol{X}_N^p(\Omega)$, thus proving the required density. $\square$

As a consequence, we have the following result.

Theorem 3.2. *The space $\boldsymbol{X}_N^p(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and there exists a constant C, such that for any $\boldsymbol{v}$ in $\boldsymbol{X}_N^p(\Omega)$:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}|\big). \tag{3.17}$$

Proof. Let $\boldsymbol{v}$ be any function in $\boldsymbol{X}_N^p(\Omega)$. Due to Lemma 3.2, there exists a sequence $(\boldsymbol{v}_k)_k$ of $\boldsymbol{W}^{1,p}(\Omega) \cap \boldsymbol{X}_N^p(\Omega)$ which converges to $\boldsymbol{v}$ in $\boldsymbol{X}^p(\Omega)$. Applying the estimate (3.4) to $\boldsymbol{v}_k$ for each k, we see that the sequence $(\boldsymbol{v}_k)_k$ is bounded in $\boldsymbol{W}^{1,p}(\Omega)$. Hence it admits a subsequence which converges weakly in $\boldsymbol{W}^{1,p}(\Omega)$ and the limit is nothing else but $\boldsymbol{v}$. The inequality (3.17) follows directly from (3.4) and it gives the continuity of the imbedding. $\square$

Remark 3.2. It is proved in Ref. 16, when the set Ω is a convex polyhedra that there is a real number $p_\Omega > 2$ such that for all p, $2 < p < p_\Omega$, any function $\boldsymbol{v}$ in $\boldsymbol{X}_N^p(\Omega)$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and satisfies the estimate (3.17) where p_Ω depends on the geometry of the domain Ω. Theorem 3.2 is an extension of this result to any p, $1 < p < \infty$ when Ω is of class $\mathcal{C}^{1,1}$.

We give now a non-compactness result of the space $\boldsymbol{X}^p(\Omega)$ into $\boldsymbol{L}^p(\Omega)$, where the proof is exactly the same in Ref. 2 for the case $p = 2$.

Proposition 3.1. *The imbedding of $\boldsymbol{X}^p(\Omega)$ into $\boldsymbol{L}^p(\Omega)$ is not compact.*

We state the following result which proves that the vanishing of the tangential component on the boundary implies the compactness. A proof in the case $p = 2$

can be found in Ref. 29. Our proof for $p \neq 2$ is based on the continuous imbedding of $\boldsymbol{X}^{\,p}_{N}(\Omega)$ in $\boldsymbol{W}^{\,1,p}(\Omega)$ and the compactness of $\boldsymbol{W}^{\,1,p}(\Omega)$ in $\boldsymbol{L}^{p}(\Omega)$.

Lemma 3.3. *The imbedding of the space $\boldsymbol{X}^{p}_{N}(\Omega)$ into $\boldsymbol{L}^{p}(\Omega)$ is compact.*

Corollary 3.2. *On the space $\boldsymbol{X}^{p}_{N}(\Omega)$, the seminorm*

$$\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{w}\|_{\mathrm{L}^{p}(\Omega)} + \sum_{\mathrm{i}=1}^{\mathrm{I}} |\langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_{\mathrm{i}}}|, \tag{3.18}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^{p}(\Omega)}$. In particular, we have the following Poincaré's inequality for every function $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ:

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^{p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{w}\|_{\mathrm{L}^{p}(\Omega)} + \sum_{\mathrm{i}=1}^{\mathrm{I}} |\langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_{\mathrm{i}}}|)$$

and the norm (3.18) is equivalent to the full norm $\|\cdot\|_{\boldsymbol{W}^{1,p}(\Omega)}$ on $\boldsymbol{X}^{p}_{N}(\Omega)$

Proof. The proof consistes in applying "Peetre-Tartar Lemma"(cf. Ref. 19, Chapter I, Theorem 2.1), with the following correspondance: $E_1 = \boldsymbol{X}^{p}_{N}(\Omega)$, $E_2 = L^{p}(\Omega)\times\boldsymbol{L}^{p}(\Omega)$, $E_3 = \boldsymbol{L}^{p}(\Omega)$, $A\boldsymbol{u} = (\mathrm{div}\,\boldsymbol{u},\, \mathbf{curl}\,\boldsymbol{u})$, $B = Id$, the identity operator. Due to the compactness result of Lemma 3.3, the canonical imbedding Id of E_1 into E_3 is compact. Besides, let $G = \boldsymbol{K}^{\,p}_{N}(\Omega)$ and $M:\ \boldsymbol{X}^{p}_{N}(\Omega) \mapsto \boldsymbol{K}^{\,p}_{N}(\Omega)$ be the following mapping: $\boldsymbol{u} \mapsto M\boldsymbol{u} = \sum_{i=1}^{I}\langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}\,\nabla q_i^{N}$. We set $\|M\boldsymbol{u}\|_{G} = \sum_{i=1}^{I}|\langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}|$. It is clear that $M \in \mathcal{L}(\boldsymbol{X}^{\,p}_{N}(\Omega),\, \boldsymbol{K}^{\,p}_{N}(\Omega))$. Next, it is clear that if $\boldsymbol{u} \in \mathrm{Ker}\,A = \boldsymbol{K}^{\,p}_{N}(\Omega)$, then $M\boldsymbol{u} = \boldsymbol{0} \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$ and this finishes the proof. $\square$

3.2. *Estimates with normal boundary conditions*

In order to prove the corresponding theorem for the space $\boldsymbol{X}^{\,p}_{T}(\Omega)$, we introduce the following linear integral operator

$$R\,\boldsymbol{\lambda}(\boldsymbol{x}) = \frac{1}{2\pi}\int_{\Gamma} \mathbf{curl}\,(\frac{\boldsymbol{\lambda}(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}) \times \boldsymbol{n}\,\mathrm{d}\sigma_{\boldsymbol{\xi}}.$$

We give some properties of this operator.

Lemma 3.4. *We have the following properties:*

(i) The operator R is compact from $\boldsymbol{L}^{p}(\Gamma)$ into $\boldsymbol{L}^{p}(\Gamma)$.

(ii) *The space* $\mathrm{Im}(Id+R)$ *is a closed subspace of* $\boldsymbol{L}^p(\Gamma)$ *and* $\mathrm{Ker}(Id+R)$ *is of finite dimension. It is spaned by the the traces of the functions* $\widetilde{\mathbf{grad}}\, q_j^T \times \boldsymbol{n}|_\Gamma$, $1 \le j \le J$, *where each* q_j^T *is the unique solution in* $W^{2,p}(\Omega^\circ)$ *of the problem*

$$\begin{cases} -\Delta q_j^T = 0 & \text{in } \Omega^\circ, \\ \partial_n q_j^T = 0 & \text{on } \Gamma, \\ \big[\, q_j^T \,\big]_k = \text{constant} \quad \text{and} \quad [\,\partial_n q_j^T\,]_k = 0, \ 1 \le k \le J, \\ \big\langle \partial_n q_j^T, 1 \big\rangle_{\Sigma_k} = \delta_{j\,k}, \ 1 \le k \le J, \end{cases} \tag{3.19}$$

(see Subsection 4.2).

(iii) *For any* $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ *we have:*

$$\|\boldsymbol{v}\times\boldsymbol{n}\|_{L^p(\Gamma)} \le C\Big(\|(Id+R)(\boldsymbol{v}\times\boldsymbol{n})\|_{L^p(\Gamma)} + \sum_{j=1}^{J} |\langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Sigma_j}|\Big). \tag{3.20}$$

Proof.

(i) According to Ref. 28, we have $R \in \mathcal{L}(\boldsymbol{L}^p(\Gamma), \boldsymbol{W}^{1,p}(\Gamma))$. The compact imbedding of the space $\boldsymbol{W}^{1,p}(\Gamma)$ in $\boldsymbol{L}^p(\Gamma)$ implies that R is compact from $\boldsymbol{L}^p(\Gamma)$ into $\boldsymbol{L}^p(\Gamma)$.

(ii) By virtue of the first point and the Fredholm alternative, we have that the space $\mathrm{Ker}(Id+R)$ is of finite dimension and $\mathrm{Im}(Id+R)$ is a closed subspace of $\boldsymbol{L}^p(\Gamma)$. We will see later in Section 4 that the functions $\widetilde{\mathbf{grad}}\, q_j^T$ belong to $\boldsymbol{W}^{1,p}(\Omega)$. Since the functions $\widetilde{\mathbf{grad}}\, q_j^T$, $1 \le j \le J$ are linearly independent, it is readily checked that $\widetilde{\mathbf{grad}}\, q_j^T \times \boldsymbol{n}|_\Gamma$ are also linearly independent for $1 \le j \le J$.
For $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$, the quantity $\boldsymbol{v}\times\boldsymbol{n}$ satisfies on Γ the following representation (see Ref. 28):

$$\begin{aligned} (Id+R)(\boldsymbol{v}\times\boldsymbol{n}) = {} & \frac{1}{2\pi}\left(\mathbf{grad}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\times\boldsymbol{n} \\ & + \frac{1}{2\pi}\left(\mathbf{grad}\int_\Gamma \frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}(\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{\xi})\,\mathrm{d}\sigma_{\boldsymbol{\xi}}\right)\times\boldsymbol{n} \\ & - \frac{1}{2\pi}\left(\mathbf{curl}\int_\Omega \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{v}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\times\boldsymbol{n}. \end{aligned} \tag{3.21}$$

As $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{W}^{1,p}(\Omega)$, this relation is still valid for $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$. Since $\widetilde{\mathbf{grad}}\, q_j^T$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$, due to (3.21), for each $1 \le j \le J$ the function $\widetilde{\mathbf{grad}}\, q_j^T \times \boldsymbol{n}$ belongs to $\mathrm{Ker}(Id+R)$. Since the dimension of $\mathrm{Ker}(Id+R)$ is J (see Ref. 28), the set $\{\widetilde{\mathbf{grad}}\, q_j^T \times \boldsymbol{n}|_\Gamma,\ 1 \le j \le J\}$ is a basis of $\mathrm{Ker}(Id+R)$.

(iii) The operator $Id+R$ is linear, continuous and surjective from $\boldsymbol{L}^p(\Gamma)$ onto $\mathrm{Im}(Id+R)$. Since $\mathrm{Ker}(Id+R)$ is of finite dimension, through the theorem of open application we deduce the existence of a constant $C>0$ such that (3.20) holds. $\square$

The result of the next theorem is a generalization of the one in Ref. 28 to the case $J\geq 1$. So, we expect that for an estimate of $\nabla\,\boldsymbol{v}$ in addition to $\operatorname{div}\boldsymbol{v}$ and $\mathbf{curl}\,\boldsymbol{v}$ the quantity $\sum_{j=1}^{J}\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}$ if $\boldsymbol{v}\cdot\boldsymbol{n}$ vanish on Γ. We skip the proof because it is similar to than of Theorem 3.1.

Theorem 3.3. *Let $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ be such that $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on Γ. Then the following estimate holds*

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\leq C(\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{j=1}^{J}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}|),\tag{3.22}$$

where the constant C depends only on p and Ω.

We give now the corresponding density result for the space $\boldsymbol{X}_T^p(\Omega)$ where the proof is exactly the same in Ref. 2 for the case $p=2$.

Lemma 3.5. *The space $\boldsymbol{W}^{1,p}(\Omega)\cap\boldsymbol{X}_T^p(\Omega)$ is dense in the space $\boldsymbol{X}_T^p(\Omega)$.*

As a consequence, the following theorem can be proved as in Theorem 3.2 by using Lemma 3.5 and Theorem 3.3.

Theorem 3.4. *The space $\boldsymbol{X}_T^p(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and for any function $\boldsymbol{v}$ in $\boldsymbol{X}_T^p(\Omega)$, we have the following estimate:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\sum_{j=1}^{J}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}|\big).\tag{3.23}$$

Remark 3.3. *We recall that if $p=2$, any function $\boldsymbol{v}$ of $\boldsymbol{H}^1(\Omega)\cap\boldsymbol{X}_T^2(\Omega)$ satisfies (see* Ref. 2, *Lemma 2.11)*

$$\|\nabla\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}=\|\mathbf{curl}\,\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}+\|\operatorname{div}\boldsymbol{v}\|^2_{L^2(\Omega)}-\int_\Gamma\mathcal{B}(\boldsymbol{v}\times\boldsymbol{n},\boldsymbol{v}\times\boldsymbol{n})\,\mathrm{d}\boldsymbol{\tau}.$$

But for $p\neq 2$, we have the inequality (3.22) *which allows us to estimate $\nabla\,\boldsymbol{v}$ by* $\mathbf{curl}\,\boldsymbol{v}$, $\operatorname{div}\boldsymbol{v}$ *and* $\sum_{j=1}^{J}|\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}|$. *As in Remark* 3.1, *we can prove the inequality* (3.9).

Corollary 3.3. *Let $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ be such that $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on Γ. Then, we have the following estimate:*

$$\|\nabla\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\operatorname{div}\boldsymbol{v}\|_{L^p(\Omega)}+\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\big).\tag{3.24}$$

Proof. The proof is similar to that of Corollary 3.1. For this, we use the result of Lemma 4.2 given in Section 4. □

We give now the following compactness result. The proof can also be found in Ref. 29 for the case $p = 2$ and our proof for $p \neq 2$ is based on the continuous imbedding of $\boldsymbol{X}^{\,p}_N(\Omega)$ in $\boldsymbol{W}^{\,1,p}(\Omega)$ and the compactness of $\boldsymbol{W}^{\,1,p}(\Omega)$ in $\boldsymbol{L}^p(\Omega)$.

Lemma 3.6. *The imbedding of $\boldsymbol{X}^p_T(\Omega)$ into $\boldsymbol{L}^{\,p}(\Omega)$ is compact.*

We skip the proof of the next corollary about equivalent norms, because it uses exactly the same tools as in the proof of Corollary 3.2.

Corollary 3.4. *On the space $\boldsymbol{X}^p_T(\Omega)$, the seminorm*

$$\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{w}\|_{\mathrm{L}^p(\Omega)} + \sum_{\mathrm{j}=1}^{\mathrm{J}} |\langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_\mathrm{j}}|, \tag{3.25}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{X}^p(\Omega)}$. In particular, we have the following Poincaré's inequality for every function $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ with $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \sum_{j=1}^{J} \langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}\big). \tag{3.26}$$

Now, in the following we show that the results of Theorem 3.4 can be extended to the case where the boundary conditions $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on Γ is replaced by inhomogeneous one. More precisely, we introduce the following space for $s \in \mathbb{R}$, $s \geq 1$:

$$\boldsymbol{X}^{\,s,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} \in W^{\,s-1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{\,s-1,p}(\Omega),\ \ \boldsymbol{v}\cdot\boldsymbol{n} \in \boldsymbol{W}^{\,s-\frac{1}{p},p}(\Gamma)\}.$$

Theorem 3.5. *The space $\boldsymbol{X}^{1,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and we have the following estimate for any $\boldsymbol{v}$ in $\boldsymbol{X}^{1,p}(\Omega)$:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big). \tag{3.27}$$

Proof. Let $\boldsymbol{v}$ be any function of $\boldsymbol{X}^{\,1,p}(\Omega)$. Due to the regularity of Ω, the following Neumann problem

$$\Delta\chi = \mathrm{div}\,\boldsymbol{v} \ \text{ in}\,\Omega \quad \text{ and } \quad \partial_n\chi = \boldsymbol{v}\cdot\boldsymbol{n} \ \text{ on}\,\Gamma,$$

has a unique solution χ in $W^{\,2,p}(\Omega)$ with the estimate

$$\|\chi\|_{W^{\,2,p}(\Omega)} \leq C\big(\|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W}^{\,1-\frac{1}{p},p}(\Gamma)}\big). \tag{3.28}$$

Hence, the function $\boldsymbol{w} = \boldsymbol{v} - \mathbf{grad}\,\chi$ is a divergence-free function of $\boldsymbol{X}^{p}_{T}(\Omega)$. Applying Theorem 3.4, we have that $\boldsymbol{w}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and then that $\boldsymbol{v}$ is in $\boldsymbol{W}^{1,p}(\Omega)$. So applying the inequality (3.10) of the point *ii)* of Remark 3.3 to $\boldsymbol{w}$ we obtain

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\big(\|\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)}\big).$$

Then, the inequality (3.27) follows directly from (3.28). □

The following result is a generalization of Theorem 3.5.

Corollary 3.5. *Let $m \in \mathbb{N}^*$ and Ω of class $\mathcal{C}^{m,1}$. Then $\boldsymbol{X}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$ and for any $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$, we have the following estimate*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(\Omega)} \le C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{m-1,p}(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{m-\frac{1}{p},p}(\Gamma)}\big). \tag{3.29}$$

Proof. We consider the same proof than the one made in the case $p = 2$ by Foias and Temam,[18]. For $m = 1$, the result is given by Theorem 3.5. To simplify the discussion, we shall write the proof for $m = 2$ and the proof is similar when $m \ge 3$. Let $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ such that $\mathrm{div}\,\boldsymbol{v} \in W^{1,p}(\Omega)$, $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{v}\cdot\boldsymbol{n} \in W^{2-1/p,p}(\Gamma)$. We already know that $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$. We set, for $i = 1, 2, 3$, $\boldsymbol{u}_i = \dfrac{\partial \boldsymbol{v}}{\partial x_i}$ and we prove that $\boldsymbol{u}_i \in \boldsymbol{W}^{1,p}(\Omega)$. We know that $\boldsymbol{u}_i \in \boldsymbol{L}^p(\Omega)$, $\mathrm{div}\,\boldsymbol{u}_i \in L^p(\Omega)$ and $\mathbf{curl}\,\boldsymbol{u}_i \in \boldsymbol{L}^p(\Omega)$. Since Ω is of class $\mathcal{C}^{2,1}$, the normal vector $\boldsymbol{n}$ can be extended to a vector field, still denoted by $\boldsymbol{n}$, such that $\boldsymbol{n} \in \mathcal{C}^{1,1}(\overline{\Omega})$. We have then:

$$\boldsymbol{u}_i\cdot\boldsymbol{n} = \frac{\partial}{\partial x_i}(\boldsymbol{v}\cdot\boldsymbol{n}) - \boldsymbol{v}\cdot\frac{\partial \boldsymbol{n}}{\partial x_i} \qquad \text{in } \Omega. \tag{3.30}$$

By the hypothesis on the normal trace of $\boldsymbol{v}$, we can consider $\boldsymbol{v}\cdot\boldsymbol{n}$ as the trace of a function in $W^{2,p}(\Omega)$ and then $\dfrac{\partial}{\partial x_i}(\boldsymbol{v}\cdot\boldsymbol{n})|_\Gamma \in W^{1-1/p,p}(\Gamma)$. Moreover, the fact that $\boldsymbol{v} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ and $\dfrac{\partial \boldsymbol{n}}{\partial x_i} \in \boldsymbol{W}^{1,\infty}(\Omega)$ implies that $\boldsymbol{v}\cdot\dfrac{\partial \boldsymbol{n}}{\partial x_i} \in W^{1-1/p,p}(\Gamma)$. So, by (3.30) $\boldsymbol{u}_i\cdot\boldsymbol{n} \in W^{1-1/p,p}(\Gamma)$. According to Theorem 3.5, $\boldsymbol{u}_i \in \boldsymbol{W}^{1,p}(\Omega)$ for $i = 1, 2, 3$. As a consequence $\boldsymbol{v}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ with the estimate (3.29). □

Using an interpolation argument, we can prove the following result.

Corollary 3.6. *Let $s = m + \sigma$, $m \in \mathbb{N}^*$ and $0 < \sigma \le 1$, Assume that Ω is of class $\mathcal{C}^{m+1,1}$. Then, the space $\boldsymbol{X}^{s,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{s,p}(\Omega)$ and for any function $\boldsymbol{v}$ in $\boldsymbol{W}^{s,p}(\Omega)$, we have the following estimate*

$$\begin{aligned}\|\boldsymbol{v}\|_{\boldsymbol{W}^{s,p}(\Omega)} \le C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{s-1,p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{s-1,p}(\Omega)} +\\ + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{s-\frac{1}{p},p}(\Gamma)}\big).\end{aligned} \tag{3.31}$$

A similar estimate can be found in Ref. 10, where $s > 3/p + 1$. In Ref. 15, we find a close result with $\boldsymbol{u}$, $\operatorname{div} \boldsymbol{u}$ and $\mathbf{curl}\, \boldsymbol{u}$ in $L^2(\Omega)$ and with $\boldsymbol{u} \times \boldsymbol{n}$ or $\boldsymbol{u} \cdot \boldsymbol{n}$ in $L^2(\Gamma)$.

4. Vector Potentials and Inf-Sup Conditions

In this section, we want to prove some results concerning the vector potentials. In Subsection 4.1, we first give a basic result about the vector potentials without boundary conditions useful for the sequel of this section (see Lemma 4.1). Next, Subsections 4.2 and 4.3 are respectively devoted to the proof of existence and uniqueness of tangential vector potentials and normal vector potentials. For the construction of these vector potentials, an important tool is the characterization of some kernels. In Subsection 4.4, we will get interested in an other type of vector potentials with vanishing trace on the boundary. Next, we present some results concerning scalar potentials and weak vector potentials.

4.1. *Vector potentials without boundary conditions*

This subsection is devoted to the proof of the following basic lemma. A detailed proof of the case $p = 2$ can be found in Lemma 3.5 of Ref. 2 or in Chapter I, Theorem 3.4. of Ref. 19, For $1 < p < \infty$, we give a different proof using the fundamental solution of the laplacian.

Lemma 4.1. *A vector field* $\boldsymbol{u}$ *in* $\boldsymbol{H}^p(\operatorname{div}, \Omega)$ *satisfies*

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in}\, \Omega \quad \text{and} \qquad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_{\mathrm{i}}} = 0, \quad 0 \leq \mathrm{i} \leq \mathrm{I}, \tag{4.1}$$

if and only if there exists a vector potential $\boldsymbol{\psi}_0$ *in* $\boldsymbol{W}^{1,\,p}(\Omega)$ *such that*

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi}_0. \tag{4.2}$$

Moreover, we can choose $\boldsymbol{\psi}_{\mathbf{0}}$ *such that* $\operatorname{div} \boldsymbol{\psi}_0 = 0$ *and we have the estimate*

$$\| \boldsymbol{\psi}_0 \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \, \| \boldsymbol{u} \|_{\boldsymbol{L}^p(\Omega)}, \tag{4.3}$$

where $C > 0$ *depends only on* p *and* Ω.

Proof.

(i) The necessity of conditions (4.1) can be established exactly with the same arguments than in Ref. 2.

(ii) Conversely, let $\boldsymbol{u}$ be any function satisfying (4.1). The idea is to extend $\boldsymbol{u}$ to the whole space so that the extended function $\widetilde{\boldsymbol{u}}$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$, is divergence-free and has a compact support. Then, it will be easy to construct its stream function by means of the fundamental solution of the Laplacian. Let then χ_0

in $W^{1,p}(\Omega)$ be the unique solution up to an additive constant of the following Neumann problem

$$-\Delta\chi_0 = 0 \ \text{ in } \Omega_0 \ \text{ and } \quad \partial_n\chi_0 = \boldsymbol{u}\cdot\boldsymbol{n} \ \text{ on } \Gamma_0, \quad \partial_n\chi_0 = 0 \quad \text{on } \partial\mathcal{O},$$

(see the introduction for the notations), and let $\chi_i \in W^{1,p}(\Omega)$ with $1 \le i \le I$, be the unique solution up to an additive constant of the problem:

$$-\Delta\chi_i = 0 \ \text{ in } \Omega_i \ \text{ and } \quad \partial_n\chi_i = \boldsymbol{u}\cdot\boldsymbol{n} \text{ on } \Gamma_i,$$

with the estimate:

$$\|\chi_i\|_{\boldsymbol{W}^{1,p}(\Omega_i)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},$$

and where $\boldsymbol{n}$ denotes the unit outward normal to Ω and $\mathcal{O}$. Now we can extend $\boldsymbol{u}$ as follows

$$\widetilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{u} & \text{in } \Omega, \\ \mathbf{grad}\,\chi_i & \text{in } \Omega_i, \quad 0 \le i \le I, \\ \mathbf{0} & \text{in } \mathbb{R}^3\backslash\bar{\mathcal{O}}. \end{cases}$$

Clearly, $\widetilde{\boldsymbol{u}}$ belongs to $\boldsymbol{H}^p(\mathrm{div}, \mathbb{R}^3)$ and is divergence-free in $\mathbb{R}^3$. The function $\boldsymbol{\psi}_0 = \mathbf{curl}(\,E * \widetilde{\boldsymbol{u}}\,)$, with E the fundamental solution of the laplacian, satisfies

$$\mathbf{curl}\,\boldsymbol{\psi}_0 = \widetilde{\boldsymbol{u}} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\psi}_0 = 0 \quad \text{in}\,\mathbb{R}^3.$$

Applying the Calderón Zygmund inequality, we obtain

$$\|\nabla\,\boldsymbol{\psi}_0\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \le C\|\Delta\,(E * \widetilde{\boldsymbol{u}})\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \le C\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Due to Proposition 2.10 of Ref. 4, $\boldsymbol{\psi}_0|_\Omega$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. As a consequence, $\boldsymbol{\psi}_0$ satisfies the condition (4.2) and the estimate (4.3). □

4.2. *Tangential vector potentials*

In this subsection we focus our attention on the construction of vector potentials in $\boldsymbol{X}^p_T(\Omega)$. We require the following preliminaries which are the equivalent to those in [2] for an arbitrary p with $1 < p < \infty$.

Lemma 4.2. *If $\boldsymbol{\psi}$ belongs to $\boldsymbol{H}^p_0(\mathrm{div}, \Omega)$, the restriction of $\boldsymbol{\psi}\cdot\boldsymbol{n}$ to any Σ_j belongs to the dual space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Sigma_j)'$, and the following Green's formula holds:*

$$\forall\chi \in W^{1,p'}(\Omega^\circ), \ \sum_{j=1}^{J}\langle\boldsymbol{\psi}\cdot\boldsymbol{n},\,[\,\chi\,]_j\rangle_{\Sigma_j} = \int_{\Omega^\circ}\boldsymbol{\psi}\cdot\mathbf{grad}\,\chi\,\mathrm{d}\boldsymbol{x} + \int_{\Omega^\circ}\chi\,\mathrm{div}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}, \quad (4.4)$$

where we recall that $[\chi]_j$ is the jump of χ through Σ_j.

We introduce the space

$$\Theta^p = \left\{r \in W^{1,p}(\Omega^\circ);\ [\,r\,]_j = \text{constant},\ 1 \le j \le J\right\}.$$

The next lemma is an extension of Lemma 3.11 in Ref. 2 to the case $1 < p < \infty$, where the proof is similar and gives a characterization of the space Θ^p.

Lemma 4.3. *Let r belong to $W^{1,p}(\Omega^\circ)$. Then r belongs to Θ^p if and only if*

$$\mathbf{curl}(\widetilde{\mathbf{grad}}\, r) = \mathbf{0} \quad \text{in}\, \Omega.$$

As shown in Proposition 3.14 of Ref. 2, the space $\boldsymbol{K}^2_T(\Omega)$ is spanned by the functions $\widetilde{\mathbf{grad}}\, q_j^T$, $1 \leq j \leq J$, where each q_j^T belongs to $H^1(\Omega^\circ)$, is unique up to an additive constant and satisfies the problem (3.19).

Corollary 4.1. *The functions $\widetilde{\mathbf{grad}}\, q_j^T$, $1 \leq j \leq J$ belong to $\boldsymbol{W}^{1,q}(\Omega)$ for any $1 < q < \infty$ and the space $\boldsymbol{K}^p_T(\Omega)$ is spanned by these functions.*

Proof. First, let us check that $\widetilde{\mathbf{grad}}\, q_j^T$ belongs to $\boldsymbol{K}^p_T(\Omega)$ for each $1 \leq j \leq J$. According to Ref. 2, the functions $\widetilde{\mathbf{grad}}\, q_j^T$ belong to $\boldsymbol{K}^2_T(\Omega)$. Then, it suffices to show that $\widetilde{\mathbf{grad}}\, q_j^T$ belong to $\boldsymbol{L}^p(\Omega)$ when $p > 2$. Observe that, thanks to Theorem 3.4, we have that $\widetilde{\mathbf{grad}}\, q_j^T$ belong to $\boldsymbol{H}^1(\Omega)$. Therefore, by using the Sobolev's imbedding, the functions $\widetilde{\mathbf{grad}}\, q_j^T$ belong to $\boldsymbol{L}^6(\Omega)$ and then to $\boldsymbol{X}^6_T(\Omega)$. It follows from Theorem 3.4 and the Sobolev's imbedding, that $\widetilde{\mathbf{grad}}\, q_j^T$ belongs to $\boldsymbol{L}^\infty(\Omega)$. As a consequence, for any $1 < q < \infty$, we have $\widetilde{\mathbf{grad}}\, q_j^T \in \boldsymbol{L}^q(\Omega)$. We deduce the first part of our statement by using again Theorem 3.4. We already know that the functions $\widetilde{\mathbf{grad}}\, q_j^T$ are linearly independent. Let us show now that those functions span $\boldsymbol{K}^p_T(\Omega)$ for any $1 < p < \infty$. Let $\boldsymbol{w} \in \boldsymbol{K}^p_T(\Omega)$. The function

$$\boldsymbol{v} = \boldsymbol{w} - \sum_{j=1}^{J} \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T.$$

belongs to $\boldsymbol{K}^p_T(\Omega)$ and satisfies $\langle \boldsymbol{v} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_k} = 0$, for $1 \leq k \leq J$. Using (3.22), we deduce that $\nabla\, \boldsymbol{v}$ is equal to zero. Then $\boldsymbol{v} = \boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{a} = \mathbf{0}$ because $\boldsymbol{a} \cdot \boldsymbol{n} = 0$ on Γ. Hence $\boldsymbol{v}$ is zero and this finish the proof. □

As in Ref. 2, we have the following result concerning tangential vector potential. We skip the proof because it is very similar to the case $p = 2$.

Theorem 4.1. *A function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\mathrm{div},\, \Omega)$ satisfies (4.1) if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that*

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega, \\ &\boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma, \quad \langle \boldsymbol{\psi} \cdot \boldsymbol{n},\, 1\rangle_{\Sigma_{\mathrm{j}}} = 0,\ 1 \leq \mathrm{j} \leq \mathrm{J}. \end{aligned} \tag{4.5}$$

This function $\boldsymbol{\psi}$ *is unique and we have the estimate:*

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.6}$$

Remark 4.1.

(i) As proved in Ref. 2, the statement of Theorem 4.1 is independent of the particular choice of the admissible set of cuts $\{\Sigma_j;\ 1 \leq j \leq J\}$.

(ii) If Ω is only Lipschitz, then $\boldsymbol{\psi} \in \boldsymbol{X}_T^p(\Omega)$ which is included in $\boldsymbol{W}^{1,p}(\Omega)$ only for some values of p.

The following result is an extension of Theorem 3.5 by using a normal trace in fractional Sobolev space.

Proposition 4.1. *Let* $0 < s < 1$. *Let*

$$\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \mathrm{div}\,\boldsymbol{v} \in L^p(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} \in W^{s-\frac{1}{p},p}(\Gamma). \tag{4.7}$$

Then $\boldsymbol{v} \in \boldsymbol{W}^{s,p}(\Omega)$ *and satisfies the estimate*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{s,p}(\Omega)} \leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{s-\frac{1}{p},p}(\Gamma)}\big). \tag{4.8}$$

Proof. Let $\boldsymbol{v}$ satisfy (4.7) and $\chi \in W^{s+1,p}(\Omega)$ a solution of the problem:

$$\Delta\chi = \mathrm{div}\,\boldsymbol{v} \quad \text{and} \quad \frac{\partial\chi}{\partial\boldsymbol{n}} = \boldsymbol{v}\cdot\boldsymbol{n} \text{ on } \Gamma.$$

We set $\boldsymbol{f} = \mathbf{curl}\,(\boldsymbol{v} - \nabla\chi)$. Then, $\boldsymbol{f}$ satisfies:

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \ \mathrm{div}\,\boldsymbol{f} = 0 \text{ in } \Omega \ \text{ and } \ \langle \boldsymbol{f}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0 \text{ for any } 1 \leq i \leq I.$$

According to Theorem 4.1, there exists a unique $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying

$$\begin{aligned} &\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{\psi} = 0 \text{ in } \Omega,\\ &\boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma, \quad \langle\boldsymbol{\psi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_\mathrm{j}} = 0, \text{ for any } 1 \leq \mathrm{j} \leq \mathrm{J}. \end{aligned}$$

Next, we set

$$\boldsymbol{z} = \boldsymbol{v} - \nabla\chi - \boldsymbol{\psi} - \sum_{j=1}^{J}\langle(\boldsymbol{v} - \nabla\chi - \boldsymbol{\psi})\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j}\,\widetilde{\mathbf{grad}}\,q_j^T.$$

Then, $\boldsymbol{z}$ belongs to $\boldsymbol{L}^p(\Omega)$ and satisfies

$$\mathbf{curl}\,\boldsymbol{z} = \mathbf{0},\ \ \mathrm{div}\,\boldsymbol{z} = 0,\ \ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma,\ \text{ and } \langle\boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0.$$

Using the characterization of the kernel $\boldsymbol{K}_T^p(\Omega)$, we deduce that $\boldsymbol{z} = 0$. This implies that $\boldsymbol{v} \in W^{s,p}(\Omega)$. The estimate (4.8) is then immediate. □

4.3. *Normal vector potentials and first elliptic problem*

This subsection is devoted to the normal vector potentials. As previously, an important tool is the characterization of the kernel $\boldsymbol{K}_N^p(\Omega)$. It is shown in Proposition 3.18 of Ref. 2, that the space $\boldsymbol{K}_N^2(\Omega)$ is spanned by the functions $\mathbf{grad}\, q_i^N$, $1 \leq i \leq I$, where each $q_i^N \in \boldsymbol{H}^1(\Omega)$ is the unique solution of the problem (3.1). Adapting the argument used in the proof of Corollary 4.1, we have the following result

Corollary 4.2. *The functions* $\mathbf{grad}\, q_i^N$ *belongs to* $\boldsymbol{W}^{1,q}(\Omega)$ *for any* $1 < q < \infty$ *and the space* $\boldsymbol{K}_N^p(\Omega)$ *is spanned by those functions for* $1 \leq i \leq I$.

Theorem 4.2. *Let* X *and* M *be two reflexive Banach spaces and* X' *and* M' *their dual spaces. Let* a *be the continuous bilinear form defined on* $X \times M$, *let* $A \in \mathcal{L}(X;\ M')$ *and* $A' \in \mathcal{L}(M;\ X')$ *be the operators defined by*

$$\forall v \in X,\ \forall w \in M,\ a(v, w) = < Av, w > = < v, A'w >$$

and $V = \operatorname{Ker} A$. *The following statements are equivalent:*

(i) There exist $\beta > 0$ *such that*

$$\inf_{\substack{w \in M \\ w \neq 0}} \sup_{\substack{v \in X \\ v \neq 0}} \frac{a(v, w)}{\|v\|_X \, \|w\|_M} \geq \beta. \tag{4.9}$$

(ii) The operator $A:\ X/V \mapsto M'$ *is an isomophism and* $1/\beta$ *is the continuity constant of* A^{-1}.

(iii) The operator $A':\ M \mapsto X' \perp V$ *is an isomophism and* $1/\beta$ *is the continuity constant of* $(A')^{-1}$.

Proof. First, we note that $ii) \Leftrightarrow iii)$ because $(X/V)' = X' \perp V$ where this last space contains the elements $f \in X'$ satisfying $\langle f,\, v\rangle = 0$ for any $v \in V$. It suffices then to prove that $i) \Leftrightarrow iii)$. We begin with the implication $i) \Rightarrow iii)$. Due to (4.9), we deduce that there exists a constant $\beta > 0$ such that:

$$\forall w \in M, \quad \|w\|_M \leq \frac{1}{\beta} \sup_{\substack{v \in X \\ v \neq 0}} \frac{|a(v,\, w)|}{\|v\|_X}.$$

So,

$$\|w\|_M \leq \frac{1}{\beta} \|A'w\|_{X'}, \tag{4.10}$$

and A' is injective. Moreover, $\operatorname{Im} A'$ is a closed subspace of X' where $A':\ M \to X'$. Moreover, $\operatorname{Im} A' = (\operatorname{Ker} A)^{\perp} = X' \perp V$. It remains to prove that $iii) \Rightarrow i)$. For this, it suffices to prove that if $iii)$ holds, then (4.10) also holds and (4.9) follows immediately. $\square$

Remark 4.2. As consequence, if the Inf-Sup condition (4.9) is satisfied, then we have the following properties:

(i) If $V = \{0\}$, then for any $f \in X'$, there exists a unique $w \in M$ such that

$$\forall v \in X,\ a(v, w) = < f, v > \quad \text{and} \quad \|w\|_M \leq \frac{1}{\beta}\|f\|_{X'}. \tag{4.11}$$

(ii) If $V \neq \{0\}$, then for any $f \in X'$, satisfying the compatibility condition:

$\forall v \in V,\ < f, v > = 0$, there exists a unique $w \in M$ such that (4.11).

(iii) For any $g \in M'$, $\exists v \in X$, unique up an additive element of V, such that:

$$\forall w \in M,\ a(v, w) = < g, w > \quad \text{and} \quad \|v\|_{X/V} \leq \frac{1}{\beta}\|g\|_{M'}.$$

Lemma 4.4. *The following Inf-Sup Condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\substack{\boldsymbol{\varphi} \in \boldsymbol{V}_T^{p'}(\Omega) \\ \boldsymbol{\varphi} \neq 0}} \sup_{\substack{\boldsymbol{\xi} \in \boldsymbol{V}_T^{p}(\Omega) \\ \boldsymbol{\xi} \neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi} \cdot \mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_T^p(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)}} \geq \beta. \tag{4.12}$$

Proof. We need the following Helmholtz decomposition: every vector function $\boldsymbol{g} \in \boldsymbol{L}^p(\Omega)$ can be decomposed into a sum $\boldsymbol{g} = \nabla\chi + \boldsymbol{z}$, where $\boldsymbol{z}$ belongs to $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ with $\mathrm{div}\,\boldsymbol{z} = 0$, χ belongs to $\boldsymbol{W}_0^{1,p}(\Omega)$ and satisfies the estimate

$$\|\nabla\chi\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.13}$$

Let $\boldsymbol{\varphi}$ any function of $\boldsymbol{V}_T^{p'}(\Omega)$. Due to Corollary 3.4 we can write

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_T^{p'}(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} = C \sup_{\substack{\boldsymbol{g} \in \boldsymbol{L}^p(\Omega) \\ \boldsymbol{g} \neq 0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}}. \tag{4.14}$$

We set

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \sum_{i=1}^{I} \langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} \nabla q_i^N,$$

so, $\widetilde{\boldsymbol{z}} \in \boldsymbol{L}^p(\Omega)$, $\mathrm{div}\,\widetilde{\boldsymbol{z}} = 0$ and $\langle \widetilde{\boldsymbol{z}} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$ for each $0 \leq i \leq I$. By Theorem 4.1, there exists a vector potential $\boldsymbol{\psi} \in \boldsymbol{V}_T^p(\Omega)$ such that $\widetilde{\boldsymbol{z}} = \mathbf{curl}\,\boldsymbol{\psi}$ in Ω. This implies that

$$\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{g}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \boldsymbol{z}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathbf{curl}\,\boldsymbol{\varphi} \cdot \widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}.$$

Moreover, we have

$$\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{i=1}^{I} |\langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i}| \|\nabla q_i^N\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + C\,\|\boldsymbol{z} \cdot \boldsymbol{n}\|_{W^{-\frac{1}{p},p}(\Gamma)}.$$

Since $\boldsymbol{z}$ belongs to $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$ and $\mathrm{div}\,\boldsymbol{z}=0$, by using the continuity of the normal trace operator on $\boldsymbol{H}^{\,p}(\mathrm{div},\,\Omega)$, (4.13) and (4.15) we obtain

$$\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)} \le C\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \le C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.15}$$

Finally, using Corollary 3.4 we can write

$$\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}} \le C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\widetilde{\boldsymbol{z}}\,\mathrm{d}\boldsymbol{x}\right|}{\|\widetilde{\boldsymbol{z}}\|_{\boldsymbol{L}^p(\Omega)}} \le C\frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{\psi}\|_{\boldsymbol{X}^p_T(\Omega)}},$$

and the Inf-Sup Condition (4.12) follows immediately from (4.14). □

In the next, we illustrate the importance goal of the Inf-Sup Condition by using it to resolve the following first elliptic system.

Proposition 4.2. *Assume that $\boldsymbol{v}$ belongs to $\boldsymbol{L}^p(\Omega)$. Then, the following problem*

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v}, \quad \mathrm{div}\,\boldsymbol{\xi}=0 & \text{in}\,\Omega,\\ \boldsymbol{\xi}\cdot\boldsymbol{n}=0,\ (\mathbf{curl}\,\boldsymbol{\xi}-\boldsymbol{v})\times\boldsymbol{n}=\boldsymbol{0} & \text{on}\,\Gamma,\\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}=0,\ 1\le j\le J, \end{cases} \tag{4.16}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)$ and we have:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.17}$$

Moreover, if $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ and Ω is of class $\mathcal{C}^{\,2,1}$, then the solution $\boldsymbol{\xi}$ is in $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{2,p}(\Omega)} \le C\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{4.18}$$

Proof.

(i) **Existence and uniqueness**. Thanks to Lemma 4.4, the following problem: find $\boldsymbol{\xi}\in\boldsymbol{V}^p_T(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\boldsymbol{V}^{p'}_T(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}. \tag{4.19}$$

satisfies the Inf-Sup condition (4.12). So, it has a unique solution $\boldsymbol{\xi}\in\boldsymbol{V}^p_T(\Omega)$ since the right-hand side defines an element of $(\,\boldsymbol{V}^{p'}_T(\Omega))'$. By Theorem 3.4, this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{\,1,p}(\Omega)$. Next, we want to extend (4.19) to any test function $\widetilde{\boldsymbol{\varphi}}$ in $\boldsymbol{X}^{p'}_T(\Omega)$. We consider the solution χ in $W^{\,1,p'}(\Omega)$ up to an additive constant of the Neumann problem:

$$\Delta\,\chi = \mathrm{div}\,\widetilde{\boldsymbol{\varphi}}\ \text{ in}\,\Omega \quad\text{and}\quad \frac{\partial\,\chi}{\partial\boldsymbol{n}}=0\ \text{ on}\,\Gamma. \tag{4.20}$$

Then, we set

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi - \sum_{j=1}^{J} \langle (\widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi)\cdot \boldsymbol{n},\, 1\rangle_{\Sigma_j}\, \widetilde{\mathbf{grad}}\, q_j^T. \tag{4.21}$$

Observe that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{V}_T^{p'}(\Omega)$. Hence (4.19) becomes: find $\boldsymbol{\xi} \in \boldsymbol{V}_T^{p}(\Omega)$ such that

$$\forall \widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{v}\cdot \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}. \tag{4.22}$$

It is easy to proof that every solution of (4.16) also solves (4.22). Conversely, let $\boldsymbol{\xi}$ the solution of the problem (4.22). Then,

$$-\Delta\,\boldsymbol{\xi} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v} \quad \text{in} \quad \Omega.$$

Moreover, since $\boldsymbol{\xi}$ belongs to the space $\boldsymbol{V}_T^{p}(\Omega)$ we have $\operatorname{div}\boldsymbol{\xi} = 0$ in Ω, $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ and $\langle \boldsymbol{\xi}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_j} = 0$ for any $1 \le j \le J$. Then, it remains to check the boundary condition $\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{v}\times\boldsymbol{n}$ on Γ of (4.16). The function $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$ belongs to $\boldsymbol{H}^{\,p}(\mathbf{curl},\Omega)$ with $\mathbf{curl}\,\boldsymbol{z} = \mathbf{0}$. Consequently, for any $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega)$ we have:

$$\int_\Omega \boldsymbol{z}\cdot \mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} \;-\; \langle \boldsymbol{z}\times\boldsymbol{n}, \widetilde{\boldsymbol{\varphi}}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} = \int_\Omega \mathbf{curl}\,\boldsymbol{z}\cdot\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\,\boldsymbol{x} = 0.$$

Using (4.22), we deduce that

$$\forall \widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_T^{p'}(\Omega), \quad \langle \boldsymbol{z}\times\boldsymbol{n}, \widetilde{\boldsymbol{\varphi}}\rangle_\Gamma = 0.$$

Let now $\boldsymbol{\mu}$ be any element of the space $\boldsymbol{W}^{1-\frac{1}{p'},p'}(\Gamma)$. So, there exists an element $\widetilde{\boldsymbol{\varphi}}$ of $\boldsymbol{W}^{1,p'}(\Omega)$ such that $\widetilde{\boldsymbol{\varphi}} = \boldsymbol{\mu}_t$ on Γ, where $\boldsymbol{\mu}_t$ is the tangential component of $\boldsymbol{\mu}$ on Γ. It is clear that $\widetilde{\boldsymbol{\varphi}}$ belongs to $\boldsymbol{X}_T^{p'}(\Omega)$ and

$$\langle \boldsymbol{z}\times\boldsymbol{n}, \boldsymbol{\mu}\rangle_\Gamma = \langle \boldsymbol{z}\times\boldsymbol{n}, \boldsymbol{\mu}_t\rangle_\Gamma = \langle \boldsymbol{z}\times\boldsymbol{n}, \widetilde{\boldsymbol{\varphi}}\rangle_\Gamma = 0.$$

This implies that $\boldsymbol{z}\times\boldsymbol{n} = \mathbf{0}$ on Γ which is the last boundary condition in (4.16).

To prove the estimate (4.17), we apply Remark 4.2 *iii)* .

(ii) **Regularity**. Now, we suppose that $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ given by the first step and $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi} - \boldsymbol{v}$. Observe that $\boldsymbol{z}$ belongs to $\boldsymbol{X}_N^{\,p}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$. This implies that $\mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$. Applying Corollary 3.5, we deduce that $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (4.18). □

Remark 4.3.

(i) Note that we can directly prove the uniqueness of the solution of the problem (4.16) by using the characterization of the kernels $\boldsymbol{K}^p_T(\Omega)$ and $\boldsymbol{K}^p_N(\Omega)$.

(ii) When $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^p(\Omega)$, then $(\mathbf{curl}\,\boldsymbol{\xi}-\boldsymbol{v})\times\boldsymbol{n}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ but neither $\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}$ nor $\boldsymbol{v}\times\boldsymbol{n}$ is defined. However, if $\boldsymbol{v}$ belongs to $\boldsymbol{H}^p(\mathbf{curl},\Omega)$, then $\boldsymbol{v}\times\boldsymbol{n}$ and $\mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}$ have a sense in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$.

With the previous proposition, the following theorem is the main result of this subsection

Theorem 4.3. *A function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\mathrm{div},\Omega)$ satisfies:*

$$\operatorname{div}\boldsymbol{u}=0\quad\text{in}\,\Omega,\ \ \boldsymbol{u}\cdot\boldsymbol{n}=0\quad\text{on}\ \Gamma\quad\text{and}\quad\langle\boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_\mathrm{j}}=0,\quad 1\le\mathrm{j}\le\mathrm{J},\tag{4.23}$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{aligned}&\boldsymbol{u}=\mathbf{curl}\,\boldsymbol{\psi}\quad\text{and}\quad\operatorname{div}\boldsymbol{\psi}=0\ \text{in}\ \Omega,\\&\boldsymbol{\psi}\times\boldsymbol{n}=\mathbf{0}\qquad\text{on}\ \Gamma,\\&\langle\boldsymbol{\psi}\cdot\boldsymbol{n},\ 1\rangle_{\Gamma_i}=0,\ \text{for any}\ 1\le i\le I.\end{aligned}\tag{4.24}$$

This function $\boldsymbol{\psi}$ is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)}\le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.\tag{4.25}$$

Proof. The necessity of conditions (4.23) can be established exactly as in [2]. The uniqueness follows from the characterization of the kernel $K^p_N(\Omega)$. Now, let us establish the existence of $\boldsymbol{\psi}$. According to Lemma 4.1, there exists $\boldsymbol{\psi}_0\in\boldsymbol{W}^{1,p}(\Omega)$ with $\operatorname{div}\boldsymbol{\psi}_0=0$ and such that $\boldsymbol{u}=\mathbf{curl}\,\boldsymbol{\psi}_0$. Due to Lemma 4.4, the following problem: find $\boldsymbol{\xi}\in\boldsymbol{V}^p_T(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\boldsymbol{V}^{p'}_T(\Omega),\ \int_\Omega\mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}=\int_\Omega\boldsymbol{\psi}_0\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}-\int_\Omega\mathbf{curl}\,\boldsymbol{\psi}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x},\tag{4.26}$$

has a unique solution $\boldsymbol{\xi}\in\boldsymbol{V}^p_T(\Omega)$. We know that due to Theorem 3.4, this solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Next, by using the same arguments as in the proof of Proposition 4.2, the problem (4.26) is in fact equivalent to

$$\begin{cases}-\Delta\,\boldsymbol{\xi}=\mathbf{0},\quad\operatorname{div}\boldsymbol{\xi}=0 & \text{in}\,\Omega,\\ \boldsymbol{\xi}\cdot\boldsymbol{n}=0,\ \mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n}=\boldsymbol{\psi}_0\times\boldsymbol{n} & \text{on}\,\Gamma,\\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}=0,\ 1\le j\le J.\end{cases}$$

Using Theorem 3.2, the function $\boldsymbol{z}=\boldsymbol{\psi}_0-\mathbf{curl}\,\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Then, the required vector potential is given by

$$\boldsymbol{\psi} = \boldsymbol{z} - \sum_{i=1}^{I} \langle \boldsymbol{z}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}\, \mathbf{grad}\, q_i^N. \qquad \square$$

Remark 4.4. If Ω is only Lipschitz, then $\boldsymbol{\psi} \in \boldsymbol{X}_N^{\,p}(\Omega)$ and the previous result holds only for some values of p.

4.4. *Other potentials*

In this subsection, we extend some results concerning vectors potentials and scalar potentials to the non hilbertian case without giving details (see Ref. 2 and Ref. 3 for the case $p = 2$). The first result is less standard, however it turns out to be useful in special cases.

Theorem 4.4. *A function $\boldsymbol{u}$ in $\boldsymbol{H}^p(\mathrm{div},\,\Omega)$ satisfies:*

$$\mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in}\,\Omega,\;\; \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on}\ \Gamma \quad \text{and} \quad \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Sigma_\mathrm{j}} = 0, \quad 1 \le \mathrm{j} \le \mathrm{J}, \tag{4.27}$$

if and only if there exists a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{aligned} &\boldsymbol{u} = \mathbf{curl}\,\boldsymbol{\psi} \quad \text{and} \quad \mathrm{div}\,(\Delta\boldsymbol{\psi}) = 0 \ \text{in}\ \Omega,\\ &\boldsymbol{\psi} = \boldsymbol{0} \qquad\quad \text{on}\ \Gamma, \qquad \langle \partial_\mathrm{n}(\mathrm{div}\,\boldsymbol{\psi}),\, 1\rangle_{\Gamma_\mathrm{i}} = 0,\ \text{for any}\ 0 \le \mathrm{i} \le \mathrm{I}. \end{aligned} \tag{4.28}$$

This function $\boldsymbol{\psi}$ is unique and we have the estimate:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Remark 4.5. When $\boldsymbol{u}$ in $\boldsymbol{W}_0^{m,p}(\Omega)$, for $m \ge 1$, satisfying (4.27), we can prove the existence of a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{W}_0^{m+1,p}(\Omega)$ satisfying $\mathrm{div}\,\Delta^{m+1}\boldsymbol{\psi} = 0$ in Ω.

Now, we give the following result concerning scalar potential.

Theorem 4.5. *Let $\boldsymbol{f} \in \boldsymbol{W}^{-m,p}(\Omega)$ for some integer $m > 0$. then the following properties are equivalent:*

(i) $\langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m,p}(\Omega)\times \boldsymbol{W}_0^{m,p'}(\Omega)} = 0$ for all $\boldsymbol{\varphi} \in \{\boldsymbol{\varphi} \in \boldsymbol{W}_0^{m,p'}(\Omega);\ \mathrm{div}\,\boldsymbol{\varphi} = 0 \ \text{in}\,\Omega\}$,

(ii) $\langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-m,p}(\Omega)\times \boldsymbol{W}_0^{m,p'}(\Omega)} = 0$ for all $\boldsymbol{\varphi} \in \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega);\ \mathrm{div}\,\boldsymbol{\varphi} = 0 \ \text{in}\,\Omega\}$,

(iii) There exists a distribution $\chi \in W^{-m+1,p}(\Omega)$, unique up to an additive constant, such that $\boldsymbol{f} = \mathbf{grad}\,\chi$ in Ω.

If in addition Ω is simply-connected, the above properties are equivalent to:

(iv) $\mathbf{curl}\,\boldsymbol{f} = \boldsymbol{0}$ *in* Ω.

We note that if Ω is not simply-connected, properties (*iii*) and (*iv*) are not equivalent. More precisely, for $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ or $\boldsymbol{f}$ in the dual space of $\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)$, we have the following result:

Theorem 4.6. *For any $\boldsymbol{f}$ in the dual space of $\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)$ with $\mathbf{curl}\,\boldsymbol{f} = \mathbf{0}$ in Ω and that satisfies*

$$\langle \boldsymbol{f},\, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{K}_T^{p'}(\Omega), \tag{4.29}$$

there exists a scalar potential χ in $L^p(\Omega)$, unique up to an additive constant, such that $\boldsymbol{f} = \mathbf{grad}\,\chi$ and the following estimate holds:

$$\|\chi\|_{L^p(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div}\,,\Omega)]'}.$$

Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, the scalar potential χ belongs to $W^{1,p}(\Omega)$ and satisfies the following estimate:

$$\|\chi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \tag{4.30}$$

Remark 4.6. If Ω is simply-connected, the condition (4.29) is empty because $\boldsymbol{K}_T^{p'}(\Omega) = \{\mathbf{0}\}$. Then, for a distribution $\boldsymbol{f}$ in the dual space $[\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega)]'$ satisfying $\mathbf{curl}\,\boldsymbol{f} = \mathbf{0}$ in Ω, there exists a unique function $\chi \in L^p(\Omega)$, up to an additive constant, such that $\boldsymbol{f} = \mathbf{grad}\,\chi$.

Now, we are interested into weak vector potentials corresponding to less regular data. We know that for a given function $\boldsymbol{f}$ in $\boldsymbol{W}^{-1,p}(\Omega)$, there exist a unique $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\Omega)$ and $\chi \in L^p(\Omega)$ such that

$$\boldsymbol{f} = -\Delta \boldsymbol{u} + \nabla\,\chi \qquad \text{and} \qquad \mathrm{div}\,\boldsymbol{u} = 0 \ \text{ in}\,\Omega, \tag{4.31}$$

and satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\chi\|_{L^p(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p}(\Omega)}.$$

Besides, the continuous imbeddings

$$[\boldsymbol{H}_0^p(\mathbf{curl},\,\Omega)]' \hookrightarrow \boldsymbol{W}^{-1/p,p}(\Omega) \quad \text{and} \quad [\boldsymbol{H}_0^p(\mathrm{div},\,\Omega)]' \hookrightarrow W^{-1/p,p}(\Omega)$$

hold. By setting $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$, we obtain the decomposition

$$\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{z} + \nabla\,\chi$$

with $\mathrm{div}\,\boldsymbol{z} = 0$ in Ω, $\boldsymbol{z}\cdot\boldsymbol{n} = 0$ on Γ. Now, since $\boldsymbol{z} \in \boldsymbol{L}^p(\Omega)$ and $\chi \in L^p(\Omega)$, we have that $\mathbf{curl}\,\boldsymbol{z} \in [\boldsymbol{H}_0^p(\mathbf{curl},\,\Omega)]'$ and $\nabla\,\chi \in [\boldsymbol{H}_0^p(\mathrm{div},\,\Omega)]'$. As a consequence

$$\boldsymbol{W}^{-1,p}(\Omega) = [\boldsymbol{H}_0^p(\mathbf{curl},\,\Omega)]' + [\boldsymbol{H}_0^p(\mathrm{div},\,\Omega)]',$$

but the sum is obviously not direct.

In fact, if $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)]'$, using the characterization of this last space, the solution $\boldsymbol{u}$ of the problem (4.31) is more regular:

Proposition 4.3. *For any* $\boldsymbol{f}$ *in the dual space* $\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)'$, *there exist a unique* $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)$ *and* $\chi \in L^p(\Omega)$ *solution to* (4.31) *that satisfy the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\chi\|_{L^p(\Omega)/\mathbb{R}} \leq C\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)]'}.$$

We next consider the following result.

Theorem 4.7. *For any* $\boldsymbol{f}$ *in the dual space of* $\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)$ *with* $\mathrm{div}\,\boldsymbol{f} = 0$ in Ω *and satisfies*

$$\langle \boldsymbol{f},\, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \tag{4.32}$$

there exists a vector potential $\boldsymbol{\xi}$ *in* $\boldsymbol{L}^p(\Omega)$, *unique up to an additive element of* $\boldsymbol{K}_T^p(\Omega)$, *such that*

$$\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}, \qquad \text{with} \qquad \mathrm{div}\,\boldsymbol{\xi} = 0 \quad \text{in}\,\Omega \quad \text{and} \quad \boldsymbol{\xi}\cdot\boldsymbol{n} = 0 \quad \text{on}\,\Gamma,$$

and such that the following estimate holds:

$$\|\boldsymbol{\xi}\|_{\boldsymbol{L}^p(\Omega)/\boldsymbol{K}_T^p(\Omega)} \leq C\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]'}.$$

Remark 4.7. If we assume that Ω has a connected boundary Γ, then condition (4.32) is empty because $\boldsymbol{K}_N^{p'}(\Omega) = \{\mathbf{0}\}$. Then, a distribution $\boldsymbol{f}$ belongs to $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'$ such that $\mathrm{div}\,\boldsymbol{f} = 0$ if and only if there exists a function $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$, such that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\xi}$, where $\mathrm{div}\,\boldsymbol{\xi} = 0$ in Ω and $\boldsymbol{\xi}\cdot\boldsymbol{n} = 0$ on Γ. Moreover, $\boldsymbol{\xi}$ is unique up to an additive element of $\boldsymbol{K}_T^p(\Omega)$.

5. The Stokes Equations with Normal Boundary Conditions

In this section we will study the following Stokes problem:

$$(\mathcal{S}_N)\ \begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{u} = 0 & \text{in}\,\Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\,\Gamma,\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases}$$

5.1. *Weak solutions*

The aim of this subsection is to give a variational formulation of problem $(\mathcal{S}_N)$ and to prove a theorem of existence and uniqueness of weak solutions. Due to the boundary conditions that we consider, the pressure is decoupled from the system. It is the reason why we are naturally reduced to solving elliptic problems which are the Stokes equations without the pressure term. We begin by proving a useful preliminary result involving Inf-Sup condition.

Lemma 5.1. *The following Inf-Sup condition holds: there exists a constant $\beta > 0$, such that*

$$\inf_{\substack{\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)\\ \boldsymbol{\varphi}\neq 0}} \sup_{\substack{\boldsymbol{\xi}\in \boldsymbol{V}_N^{p}(\Omega)\\ \boldsymbol{\xi}\neq 0}} \frac{\int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\xi}\|_{\boldsymbol{X}_N^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)}} \geq \beta. \tag{5.1}$$

Proof. The proof is very similar to that of Lemma 4.4. Let $\boldsymbol{\varphi}$ be any function of $\boldsymbol{V}_N^{p'}(\Omega)$. Due to Corollary 3.2, we can write: for any $\boldsymbol{\varphi}\in \boldsymbol{V}_N^{p'}(\Omega)$

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}_N^{p'}(\Omega)} \leq C\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} = C\sup_{\substack{\boldsymbol{g}\in\boldsymbol{L}^p(\Omega)\\ \boldsymbol{g}\neq 0}} \frac{\left|\int_\Omega \mathbf{curl}\,\boldsymbol{\varphi}\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}\right|}{\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}}.$$

We use now the Helmholtz decomposition $\boldsymbol{g} = \nabla\chi + \boldsymbol{z}$, where $\chi\in W^{1,p}(\Omega)$ and $\boldsymbol{z}$ belongs to $\boldsymbol{H}^p(\mathrm{div},\Omega)$ with $\mathrm{div}\,\boldsymbol{z} = 0$ in Ω and $\boldsymbol{z}\cdot\boldsymbol{n} = 0$ on Γ. Moreover, we have the estimate

$$\|\nabla\chi\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{L}^p(\Omega)}.$$

We set

$$\widetilde{\boldsymbol{z}} = \boldsymbol{z} - \sum_{j=1}^{J}\langle \boldsymbol{z}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\,\widetilde{\mathbf{grad}}\,q_j^T,$$

and we use Theorem 4.3. □

Proposition 5.1. *Let $\boldsymbol{f}\in(\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega))'$ with $\mathrm{div}\,\boldsymbol{f} = 0$ in Ω and satisfying the compatibility condition:*

$$\forall\boldsymbol{v}\in\boldsymbol{K}_N^{p'}(\Omega),\qquad \langle\boldsymbol{f},\,\boldsymbol{v}\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} = 0. \tag{5.2}$$

Then, the following problem

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = \boldsymbol{f} \quad\text{and}\quad \mathrm{div}\,\boldsymbol{\xi} = 0 & \text{in}\,\Omega,\\ \boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma,\\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0,\ 1\leq i\leq I, \end{cases} \tag{5.3}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)$ and we have:

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'}. \tag{5.4}$$

Proof. Due to Theorem 5.1 and next to Theorem 3.2, the problem: find $\boldsymbol{\xi}\in\boldsymbol{V}_N^p(\Omega)$ such that

$$\forall\boldsymbol{\varphi}\in\boldsymbol{V}_N^{p'}(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega, \tag{5.5}$$

has a unique solution $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$. Let us $\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_N^{p'}(\Omega)$ and we consider the solution χ in $W_0^{1,p'}(\Omega)$ of $\Delta\,\chi = \operatorname{div}\widetilde{\boldsymbol{\varphi}}$ in Ω. Setting

$$\boldsymbol{\varphi} = \widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi - \sum_{i=1}^{I}\langle(\widetilde{\boldsymbol{\varphi}} - \mathbf{grad}\,\chi)\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i}\,\mathbf{grad}\,q_i^N,$$

we observe that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{V}_N^{p'}(\Omega)$ and using (5.2) problem (5.5) becomes: find $\boldsymbol{\xi} \in \boldsymbol{V}_N^{p}(\Omega)$ such that

$$\forall\widetilde{\boldsymbol{\varphi}} \in \boldsymbol{X}_N^{p'}(\Omega),\ \int_\Omega \mathbf{curl}\,\boldsymbol{\xi}\cdot\mathbf{curl}\,\widetilde{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x} = \langle\boldsymbol{f},\,\widetilde{\boldsymbol{\varphi}}\rangle_\Omega. \tag{5.6}$$

We check that the problems (5.3) and (5.6) are equivalent and we deduce the compatibility condition (5.2) from (5.6). We may also apply Remark 4.2 *iii*) in order to prove the estimate (5.4). □

Remark 5.1.

(i) Thanks to the characterization of the kernels $\boldsymbol{K}_T^p(\Omega)$ and $\boldsymbol{K}_N^p(\Omega)$, we can in fact show directly the uniqueness of the solution $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ of problem (5.3).

(ii) We can replace in (5.3) the right hand side by the **curl** of an element $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$. Indeed, due to Theorem 4.7, every element $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega])'$ with $\operatorname{div}\boldsymbol{f} = 0$ in Ω and satisying the compatibility condition (5.2), can be written as the **curl** of a function $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$.

(iii) Observe that, by using (5.8) below, the problem (5.3) is equivalent to: find $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta\,\boldsymbol{\xi} = \mathbf{curl}\,\boldsymbol{v} & \text{in}\,\Omega,\\ \boldsymbol{\xi}\times\boldsymbol{n} = 0 \quad\text{and}\quad \dfrac{\partial\,\boldsymbol{\xi}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} - 2K\boldsymbol{\xi}\cdot\boldsymbol{n} = 0 & \text{on}\,\Gamma,\\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0,\ 1\le i\le I, & \end{cases}$$

where the second boundary condition is a Fourier-Robin type boundary condition.

Now, we consider the case of inhomogeneous boundary condition.

Corollary 5.1. *Let* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *with:* $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$, $\operatorname{div}\boldsymbol{f} = 0$ *in* Ω *and satisfying the compatibility condition* (5.2) *and* $\boldsymbol{g}\times\boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. *Then, the following problem*

$$(E_N)\qquad \begin{cases} -\Delta\,\boldsymbol{\xi} = \boldsymbol{f} \quad\text{and}\quad \operatorname{div}\boldsymbol{\xi} = 0 & \text{in}\,\Omega,\\ \boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} & \text{on}\,\Gamma,\\ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0,\ 1\le i\le I, & \end{cases}$$

has a unique solution in $\boldsymbol{W}^{1,p}(\Omega)$ *and we have:*

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\Big(\|\,\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]'} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\Big). \tag{5.7}$$

Proof. Let $\boldsymbol{\xi}_0 \in \boldsymbol{W}^{\,1,p}(\Omega)$ be the divergence free lift of $\boldsymbol{g}$: $\boldsymbol{\xi}_0 = \boldsymbol{g}_t$ on Γ, $\operatorname{div}\boldsymbol{\xi}_0 = 0$ in Ω with the estimate

$$\|\boldsymbol{\xi}_0\|_{\boldsymbol{W}^{\,1,p}(\Omega)} \leq C\|\boldsymbol{g}_t\|_{\boldsymbol{W}^{\,1-1/p,p}(\Gamma)}.$$

Next, observe that $\boldsymbol{F} = \boldsymbol{f} - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\xi}_0$ belongs to $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$ and satisfies the compatibility condition (5.2). □

Now, we define the space

$$\boldsymbol{Z}^{\,p}(\Omega) = \left\{\boldsymbol{v} \in\, \boldsymbol{W}^{\,1,p}(\Omega);\ \operatorname{div}\boldsymbol{v} \in W^{\,1,p}(\Omega)\right\},$$

which is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{Z}^{\,p}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\operatorname{div}\boldsymbol{v}\|_{W^{1,p}(\Omega)}.$$

We verify that the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{Z}^{\,p}(\Omega)$. The following result is proved in the cas $p = 2$ by Heron,[22] for the functions of $\boldsymbol{H}^2(\Omega)$.

Lemma 5.2. *Assume that* Ω *is of class* $\mathcal{C}^{\,2,1}$. *Every function* $\boldsymbol{v} \in \boldsymbol{Z}^p(\Omega)$ *satisfies:*

$$\operatorname{div}\boldsymbol{v} = \operatorname{div}_\Gamma \boldsymbol{v_t} - 2K\boldsymbol{v}\cdot\boldsymbol{n} + \frac{\partial\,\boldsymbol{v}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} \quad \text{in } W^{\,-1/p,p}(\Gamma) \tag{5.8}$$

where K *denotes the mean curvature of* Γ, $\boldsymbol{v_t} = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ *is the tangential component of* $\boldsymbol{v}$ *and* $\operatorname{div}_\Gamma$ *is the surface divergence. In particular, the following mapping*

$$\boldsymbol{v} \mapsto \frac{\partial\,\boldsymbol{v}}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} \ \text{ from } \boldsymbol{Z}^p(\Omega) \text{ into } W^{\,-1/p,p}(\Gamma)$$

is continuous.

Proof. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$. Then,

$$\operatorname{div}\boldsymbol{v} = \sum_{j=1}^{2}\frac{\partial\,v_j}{\partial\,s_j} + \sum_{j,k=1}^{2} v_k\,\frac{\partial\,\boldsymbol{\tau}_k}{\partial\,s_j}\cdot\boldsymbol{\tau}_j + \boldsymbol{v}_n\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{n}}{\partial\,s_j}\cdot\boldsymbol{\tau}_j + \sum_{k=1}^{2} v_k\frac{\partial\,\boldsymbol{\tau}_k}{\partial\,\boldsymbol{n}}\cdot\boldsymbol{n} + \frac{\partial\,v_n}{\partial\,\boldsymbol{n}}\,,$$

i.e. $\boldsymbol{v}$ satisfies the formula (5.8) with $K = -\frac{1}{2}\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{n}}{\partial\,s_j}\cdot\boldsymbol{\tau}_j$.

Now, let $\boldsymbol{v}$ be any function in $\boldsymbol{Z}^{\,p}(\Omega)$. Since $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{Z}^{\,p}(\Omega)$, there exists a sequence $(\boldsymbol{v}_k)_k$ of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ which converges to $\boldsymbol{v}$ in $\boldsymbol{Z}^{\,p}(\Omega)$ and we have the relation:

$$\operatorname{div} \boldsymbol{v}_k = \operatorname{div}_\Gamma (\boldsymbol{v}_k)_{\boldsymbol{t}} - 2K \boldsymbol{v}_k \cdot \boldsymbol{n} + \frac{\partial \boldsymbol{v}_k}{\partial \boldsymbol{n}} \cdot \boldsymbol{n} \quad \text{on } \Gamma. \tag{5.9}$$

Hence, $\operatorname{div} \boldsymbol{v}_k \to \operatorname{div} \boldsymbol{v}$ in $W^{\,1,p}(\Omega)$ implies that $\operatorname{div} \boldsymbol{v}_k \to \operatorname{div} \boldsymbol{v}$ in $\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$. Next, we have $(\boldsymbol{v}_k)_{\boldsymbol{t}} \to \boldsymbol{v}_{\boldsymbol{t}}$ in $\boldsymbol{W}^{\,1-\frac{1}{p},p}(\Gamma)$, since $\boldsymbol{v}_k \to \boldsymbol{v}$ in $\boldsymbol{W}^{\,1,p}(\Omega)$. As a consequence, $\operatorname{div}_\Gamma (\boldsymbol{v}_k)_{\boldsymbol{t}} \to \operatorname{div}_\Gamma \boldsymbol{v}_{\boldsymbol{t}}$ in $W^{\,-\frac{1}{p},p}(\Gamma)$. Moreover, since the domain Ω is of class $\mathcal{C}^{\,2,1}$, then $K \in W^{\,1,\infty}(\Omega)$. This implies that $2K\boldsymbol{v}_k \cdot \boldsymbol{n} \to 2K\boldsymbol{v} \cdot \boldsymbol{n}$ in $\boldsymbol{W}^{\,1-\frac{1}{p},p}(\Gamma)$, Finally, by passing to the limit $k \to +\infty$ in (5.9), we deduce the convergence of the terme $\dfrac{\partial \boldsymbol{v}_k}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}$ in $W^{\,-\frac{1}{p},p}(\Gamma)$ to an element wich will also be denoted by $\dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}$. This yields immediately (5.8). □

Similarly, we define the space

$$\boldsymbol{Y}^{\,p}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{\,1,p}(\Omega);\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{\,1,p}(\Omega) \right\},$$

which is a Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{Y}^{\,p}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

We verify that the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\boldsymbol{Y}^{\,p}(\Omega)$ and as previously, we can prove that the following formula holds for any $\boldsymbol{v} \in \boldsymbol{Y}^{p}(\Omega)$:

$$\mathbf{curl}\, \boldsymbol{v} = \sum_{j=1}^{2} \frac{\partial \boldsymbol{v}}{\partial s_j} \times \boldsymbol{\tau}_j + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \times \boldsymbol{n} \qquad \text{in } \boldsymbol{W}^{-1/p,p}(\Gamma). \tag{5.10}$$

In particular, the following mapping:

$$\boldsymbol{v} \longmapsto \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \times \boldsymbol{n} \ \text{ from } \boldsymbol{Y}^{\,p}(\Omega) \text{ into } \boldsymbol{W}^{\,-1/p,p}(\Gamma) \tag{5.11}$$

is continuous .

We give the following corollary which extends Theorem 3.2 to the case where the boundary condition $\boldsymbol{v} \times \boldsymbol{n} = \mathbf{0}$ on Γ is replaced by inhomogeneous one. We introduce the following space for $s \in \mathbb{R}$, $s \geq 1$:

$$\boldsymbol{Y}^{\,s,p}(\Omega) = \Big\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega); \operatorname{div} \boldsymbol{v} \in W^{\,s-1,p}(\Omega),\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{W}^{\,s-1,p}(\Omega), \boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{\,s-\frac{1}{p},p}(\Gamma) \Big\}.$$

Corollary 5.2. *The space $\boldsymbol{Y}^{1,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{1,p}(\Omega)$ and we have the following estimate: for any $\boldsymbol{v}$ in $\boldsymbol{Y}^{1,p}(\Omega)$,*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\operatorname{div} \boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{v} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} \Big). \tag{5.12}$$

Proof. Let $\boldsymbol{v}$ be any function of $\boldsymbol{Y}^{1,p}(\Omega)$. We set $\boldsymbol{z} = \boldsymbol{v} - \mathbf{curl}\,\boldsymbol{\xi}$ where $\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ is the solution of the problem (4.16). Hence, $\boldsymbol{z}$ belongs to the space $\boldsymbol{X}_N^p(\Omega)$. By Theorem 3.2 and (3.10), $\boldsymbol{z}$ even belongs to $\boldsymbol{W}^{1,p}(\Omega)$ with the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{z}\|_{L^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)}\big). \tag{5.13}$$

Then, it suffices to prove that $\mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega)$ in order to obtain $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$. We set $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{\xi}$. Then $\boldsymbol{w}$ satisfies

$$\begin{cases} \Delta\,\boldsymbol{w} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \quad \text{and} \quad \mathrm{div}\,\boldsymbol{w} = 0 \ \text{ in } \Omega \\ \boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{n} \ \text{ on } \Gamma \quad \text{and} \quad \langle \boldsymbol{w}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0. \end{cases}$$

Since $\mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{v} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, due to Corollary 5.1, the function $\boldsymbol{w}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq \big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big) \tag{5.14}$$

and the inequality (5.12) can be deduced by using inequalities (5.13) and (5.14). □

More generally, we have:

Corollary 5.3.

(i) *Let $m \in \mathbb{N}^*$ and Ω is of class $\mathcal{C}^{m,1}$. Then the space $\boldsymbol{Y}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$ and we have the following estimate: for any function $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$,*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(\Omega)} \leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{m-1,p}(\Omega)} + \\ + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{m-\frac{1}{p},p}(\Gamma)}\Big).$$

(ii) *Let $s = m+\sigma$, $m \in \mathbb{N}^*$ and $0 < \sigma \leq 1$, Assume that Ω is of class $\mathcal{C}^{m+1,1}$. Then, the space $\boldsymbol{Y}^{s,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{s,p}(\Omega)$ and for any function $\boldsymbol{v}$ in $\boldsymbol{Y}^{m,p}(\Omega)$ we have the following estimate:*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{s,p}(\Omega)} \leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{s-1,p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{s-1,p}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{s-\frac{1}{p},p}(\Gamma)}\Big).$$

Proof.

(i) In order to simplify the discussion, we shall write the proof for $m = 2$. For $m = 1$, the result is given by Corollary 5.2 and then the proof is similar when $m \geq 3$. We already know that $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{v}$ satisfying the estimate (5.15) with $m = 1$. Using formula (5.8), we obtain directly that $\dfrac{\partial\,\boldsymbol{v}}{\partial\,\boldsymbol{n}} \cdot \boldsymbol{n} \in W^{1-1/p,p}(\Gamma)$ and we have the estimate:

$$\|\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \cdot \boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma)} \leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \\ + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{1,p}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma)}\Big). \quad (5.15)$$

Next, we have

$$\big(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\big)_t = v_n \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{n}} + \frac{\partial \boldsymbol{v}_t}{\partial \boldsymbol{n}}.$$

Since $v_n \in W^{1-1/p,p}(\Gamma)$ and $\boldsymbol{v}_t \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, we deduce by the regularity assumption on Ω that $\big(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\big)_t \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ and we have the estimate:

$$\|\big(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\big)_t\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} \leq C\big(\|v_n\|_{W^{1-1/p,p}(\Gamma)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}\big), \quad (5.16)$$
$$\leq C\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{1,p}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma)}\big).$$

As a consequence, $\dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}$ belongs to $\boldsymbol{W}^{1-1/p,p}(\Gamma)$. Moreover, since $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$, we have that $\Delta\,\boldsymbol{v} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} - \nabla\,\mathrm{div}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$. Using the regularity resuls for the Neumann problem, we deduce that $\boldsymbol{v}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and we have the estimate:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\big(\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{1,p}(\Omega)} + \|\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\big) \quad (5.17)$$

The desired estimate in the point $i)$ can be obtained from (5.15) and (5.16) and (5.17).

(ii) This point can be proved by using a simple interpolation argument. □

As a consequence, we have the following regularity result.

Corollary 5.4. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ satisfying the compatibility condition* (5.2)*, then the solution $\boldsymbol{\xi}$ given by Proposition 5.1 belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate:*

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\|\,\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \quad (5.18)$$

Proof. We set $\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{\xi}$. Then, the function $\boldsymbol{z}$ satisfies:

$$\boldsymbol{z} \in \boldsymbol{L}^p(\Omega),\ \ \mathbf{curl}\,\boldsymbol{z} = \boldsymbol{f} \in \boldsymbol{L}^p(\Omega),\ \ \mathrm{div}\,\boldsymbol{z} = 0 \ \text{ in } \Omega \ \text{ and } \ \boldsymbol{z}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma.$$

Due to Theorem 3.4, $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. As a consequence $\boldsymbol{\xi}$ satisfies:

$$\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega),\ \ \mathbf{curl}\,\boldsymbol{\xi} \in \boldsymbol{W}^{1,p}(\Omega),\ \ \mathrm{div}\,\boldsymbol{\xi} = 0 \ \text{ in } \Omega \ \text{ and } \ \boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{0} \ \text{ on } \Gamma.$$

We deduce from Corollary 5.3 that the solution $\boldsymbol{\xi}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate (5.18). □

Using an interpolation argument, we deduce the following theorem.

Theorem 5.1. *Let s be a real number such that $0 \leq s \leq 1$. Let $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in \boldsymbol{W}^{s,p}(\Omega)$. Then, problem* (5.3) *has a unique solution $\boldsymbol{\xi} \in \boldsymbol{W}^{1+s,p}(\Omega)$ satisfying the estimate*

$$\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{1+s,p}(\Omega)} \leq C\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{s,p}(\Omega)}.$$

The next theorem provides the information on the solvability, in weak sense, of the Stokes problem $(\mathcal{S}_N)$.

Theorem 5.2. **(Weak solutions for $(\mathcal{S}_{\mathrm{N}})$)** *Let $\boldsymbol{f}$, $\boldsymbol{g}$, π_0 with*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]', \;\; \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \;\; \pi_0 \in W^{\,1-1/p,p}(\Gamma), \tag{5.19}$$

satisfying the compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega - \int_\Gamma \pi_0\, \boldsymbol{v}\cdot\boldsymbol{n}\, \mathrm{d}\boldsymbol{s} = 0, \tag{5.20}$$

where $\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathbf{curl}\,,\Omega)}$. Then, the Stokes problem $(\mathcal{S}_N)$ has a unique solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \; W^{\,1,p}(\Omega)$ satisfying the estimate

$$\begin{aligned}\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\,\pi\,\|_{W^{\,1,p}(\Omega)} \leq C\big(\|\,\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} +\\ + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\big).\end{aligned} \tag{5.21}$$

Proof. First, we consider the problem

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \quad \text{in}\,\Omega, \qquad \pi = \pi_0 \quad \text{on}\,\Gamma.$$

Because $\operatorname{div}\boldsymbol{f} \in W^{\,-1,p}(\Omega)$, this problem has a unique solution $\pi \in W^{\,1,p}(\Omega)$ satisfying the estimate

$$\|\pi\|_{W^{\,1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega)]'} + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\big). \tag{5.22}$$

Next, because $\nabla\,\pi \in \boldsymbol{L}^p(\Omega)$, observe that $\boldsymbol{F} = \boldsymbol{f} - \nabla\,\pi$ is an element of the dual space $[\boldsymbol{H}_0^{\,p'}(\mathbf{curl},\,\Omega)]'$ and satisfies the compatibility condition (5.2). So problem $(\mathcal{S}_N)$ becomes: $-\Delta\,\boldsymbol{u} = \boldsymbol{F}$ in Ω, $\operatorname{div}\boldsymbol{u} = 0$ in Ω, $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ on Γ and $\langle\boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0$ for any $1 \leq i \leq I$ which is equivalent to: Find $\boldsymbol{u} \in \boldsymbol{X}_N^{\,p}(\Omega)$ such that:

$$\boldsymbol{u} - \boldsymbol{\xi}_0 \in \boldsymbol{V}_N^{\,p}(\Omega)$$

$$\forall \boldsymbol{v} \in \boldsymbol{V}_N^{\,p'}(\Omega), \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \langle\,\boldsymbol{F}\,,\;\boldsymbol{v}\,\rangle_\Omega,$$

where $\boldsymbol{\xi}_0$ is the function given in the proof of Corollary 5.1. The compatibility condition (5.20) comes from the last variational formulation by taking $\boldsymbol{v} \in \boldsymbol{K}_N^{\,p'}(\Omega)$.

Moreover, since $\boldsymbol{F}$ satisfies the assumptions of Corollary 5.1, this problem has a unique solution $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate

$$\| \boldsymbol{u} \|_{W^{1,p}(\Omega)} \leq C\big(\| \boldsymbol{F} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big). \tag{5.23}$$

Finally, the pair $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ is the unique solution of the problem $(\mathcal{S}_N)$ and the estimate (5.21) follows easily from (5.22) and (5.23). □

Remark 5.2. If we take $\pi_0 \in W^{-1/p,p}(\Gamma)$, we obtain that $\pi \in L^p(\Omega)$ a unique solution of the problem:

$$-\Delta\, \pi = \operatorname{div} \boldsymbol{f} \quad \text{in } \Omega \quad \text{and} \quad \pi = \pi_0 \quad \text{on } \Gamma.$$

But we are not able to solve problem $(\mathcal{S}_N)$ because, in this case, $\boldsymbol{f} = \mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{u}) + \nabla\, \pi \notin [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'$.

More generally, we can study the following Stokes problem when the divergence operator does not vanish and it is a given function:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \;\text{ and }\; \operatorname{div} \boldsymbol{u} = \chi & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\, \Gamma, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases} \tag{5.24}$$

Corollary 5.5. *Let* $\boldsymbol{f}$, χ, $\boldsymbol{g}$, π_0 *with*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]', \quad \chi \in W^{1,p}(\Omega), \quad \boldsymbol{g} \times \boldsymbol{n} \in W^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{1-1/p,p}(\Gamma)$$

and satisfying the compatibility condition:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v}\rangle_\Omega - \int_\Gamma (\pi_0 - \chi)\, \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{s} = 0. \tag{5.25}$$

Then, the Stokes problem (5.24) *has exactly one solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ *and* $\pi \in W^{1,p}(\Omega)$*. Moreover, there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \leq C\, \Big(\| \boldsymbol{f} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\chi\|_{W^{1,p}(\Omega)} + \\ + \| \boldsymbol{g} \|_{W^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\Big). \tag{5.26}$$

Remark 5.3. For the same reason as in Remark 5.2, we can not suppose that $\chi \in L^p(\Omega)$ only.

5.2. *Strong solutions and regularity for the Stokes system* $(\mathcal{S}_N)$

In this subsection, we propose to study the question of the regularity of the solutions of problem $(\mathcal{S}_N)$, when the data are more regular. We need the following preliminary result.

Lemma 5.3. *The mapping* $\boldsymbol{v} \longmapsto \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}$ *is continuous from* $\boldsymbol{W}^{1,p}(\Omega)$ *into* $W^{-1/p,p}(\Gamma)$ *and we have the relation:*

$$\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n} = \big(\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{v}_t}{\partial\,s_j}\times\boldsymbol{\tau}_j\big)\cdot\boldsymbol{n}\quad\text{on}\,\Gamma,\ \text{ in the sense of } W^{-1/p,p}(\Gamma). \tag{5.27}$$

If moreover, $\boldsymbol{v}\times\boldsymbol{n}\in\boldsymbol{W}^{2-1/p,p}(\Gamma)$*, then* $\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}\in W^{1-1/p,p}(\Gamma)$ *with the estimate:*

$$\|\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{1-1/p,p}(\Gamma)} \leq C\|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)}.$$

Proof. Let $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$. By the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\boldsymbol{W}^{1,p}(\Omega)$, there exists a sequence $\boldsymbol{v}_k\in\boldsymbol{\mathcal{D}}(\overline{\Omega})$ which converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{1,p}(\Omega)$. Using (5.10), we deduce that

$$\mathbf{curl}\,\boldsymbol{v}_k\cdot\boldsymbol{n} = \big(\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{v}_k}{\partial\,s_j}\times\boldsymbol{\tau}_j\big)\cdot\boldsymbol{n}\quad\text{on}\,\Gamma.$$

Since $\boldsymbol{v}_k$ converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{1-1/p,p}(\Gamma)$, we deduce that the term

$$\big(\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{v}_k}{\partial\,s_j}\times\boldsymbol{\tau}_j\big)\cdot\boldsymbol{n}\ \text{ converges to }\ \big(\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{v}}{\partial\,s_j}\times\boldsymbol{\tau}_j\big)\cdot\boldsymbol{n}\ \text{ in }\ W^{-1/p,p}(\Gamma).$$

Moreover, $\mathbf{curl}\,\boldsymbol{v}$ belongs to $\boldsymbol{H}^{p}(\mathrm{div},\Omega)$ and by the continuity of the normal trace operator, we have the convergence of $\mathbf{curl}\,\boldsymbol{v}_k\cdot\boldsymbol{n}$ to $\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}$ in $W^{-1/p,p}(\Gamma)$, which proves the following formula:

$$\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n} = \Big(\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{v}}{\partial\,s_j}\times\boldsymbol{\tau}_j\Big)\cdot\boldsymbol{n}\quad\text{on}\,\Gamma. \tag{5.28}$$

Consequently,

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n} = {}& \Big(\sum_{j=1}^{2}\frac{\partial\,\boldsymbol{v}_t}{\partial\,s_j}\times\boldsymbol{\tau}_j\Big)\cdot\boldsymbol{n} + \Big(\sum_{j=1}^{2}\frac{\partial\,(\boldsymbol{v}\cdot\boldsymbol{n})}{\partial\,s_j}\boldsymbol{n}\times\boldsymbol{\tau}_j\Big)\cdot\boldsymbol{n} + \\ & + \Big(\sum_{j=1}^{2}(\boldsymbol{v}\cdot\boldsymbol{n})\frac{\partial\,\boldsymbol{n}}{\partial\,s_j}\times\boldsymbol{\tau}_j\Big)\cdot\boldsymbol{n}.\end{aligned} \tag{5.29}$$

Observe that the two last terms vanish. Moreover, since $\boldsymbol{v}\times\boldsymbol{n}\in\boldsymbol{W}^{2-1/p,p}(\Gamma)$, the tangential derivation on Γ of $\boldsymbol{v}_t$ belongs to $\boldsymbol{W}^{1-1/p,p}(\Gamma)$. Thanks to the regularity of Γ, the first term in (5.29) belongs to $\boldsymbol{W}^{1-1/p,p}(\Gamma)$. This prove Lemma 5.3. $\square$

Theorem 5.3. **(Strong solutions for $(\mathcal{S}_{\rm N})$)** *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 with:*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma), \;\; \pi_0 \in W^{1-1/p,p}(\Gamma) \tag{5.30}$$

satisfying the compatibility condition (5.20). *Then, the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ of the Stokes problem $(\mathcal{S}_N)$ given by Theorem 5.2 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfying the estimate:*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)} \le C\Big(\| \boldsymbol{f} \|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g} \times \boldsymbol{n}\|_{\boldsymbol{W}^{2-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\Big).$$

Proof. Let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ be the solution given by Theorem 5.2. It suffices to prove that $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. The function $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$ satisfies:

$$\boldsymbol{z} \in \boldsymbol{L}^p(\Omega), \;\; \operatorname{div} \boldsymbol{z} = 0, \quad \mathbf{curl}\, \boldsymbol{z} \in \boldsymbol{L}^p(\Omega).$$

Moreover, since $\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$ and due to Lemma 5.3, $\boldsymbol{z} \cdot \boldsymbol{n}$ belongs to $W^{1-1/p,p}(\Gamma)$. We deduce from Theorem 3.5 that $\boldsymbol{z} \in \boldsymbol{W}^{1,p}(\Omega)$. As a consequence, it follows from Corollary 3.5 that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$. $\square$

We can also consider strong solutions in the case when the divergence operator does not vanish and we have only to consider regular boundary data for the velocity. So, the proof of the following result is quite similar to that of Theorem 5.3 above.

Corollary 5.6. *Let $\boldsymbol{f}$, $\boldsymbol{g}$, χ, π_0 with:*

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma), \;\; \chi \in W^{1,p}(\Omega), \quad \pi_0 \in W^{1-1/p,p}(\Gamma) \tag{5.31}$$

satisfying the compatibility condition (5.25). *Then, the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ of the Stokes problem* (5.24) *given by Corollary 5.5 belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ with the corresponding estimate.*

5.3. *Very weak solutions for the Stokes system* $(\mathcal{S}_N)$

In this subsection, we are going to study the existence of very weak solutions for the Stokes problem $(\mathcal{S}_N)$. Before, we give some preliminary results.

We introduce the space:

$$\boldsymbol{N}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(\Omega); \; \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{H}_0^p(\mathbf{curl}, \Omega)\},$$

equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{N}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\mathbf{curl}\, \boldsymbol{v}\|_{\boldsymbol{H}^p(\mathbf{curl},\, \Omega)}.$$

It is easy to verify that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{N}^p(\Omega)$.

We introduce also the following spaces

$$\boldsymbol{L}^p_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \operatorname{div} \boldsymbol{v} = 0\} \quad \text{and} \quad \boldsymbol{G}^{\,p}(\Omega) = \{\nabla\theta;\ \theta \in W_0^{1,p}(\Omega)\}.$$

It is clear that $\boldsymbol{L}^p_\sigma(\Omega)$ is a closed subspace of $\boldsymbol{L}^p(\Omega)$. Poincaré's inequality implies that it is the same for $\boldsymbol{G}^p(\Omega)$. The following lemma gives a characterization for the dual space of $\boldsymbol{L}^p_\sigma(\Omega)$.

Lemma 5.4. *We have the following properties:*

(i) $\boldsymbol{L}^p(\Omega) = \boldsymbol{L}^p_\sigma(\Omega) \oplus \boldsymbol{G}^{\,p}(\Omega)$.

(ii) $(\boldsymbol{L}^p_\sigma(\Omega))' = \boldsymbol{L}^{p'}_\sigma(\Omega)$.

Proof.

(i) It is clear that $\boldsymbol{L}^p_\sigma(\Omega) \cap \boldsymbol{G}^p(\Omega) = \{\boldsymbol{0}\}$. Let $\boldsymbol{v}$ be any element of $\boldsymbol{L}^p(\Omega)$ and $\chi \in W_0^{1,p}(\Omega)$ satisfying: $\Delta\,\chi = \operatorname{div} \boldsymbol{v}$ in Ω. Setting $\boldsymbol{u} = \boldsymbol{v} - \nabla\,\chi$, we deduce the point i).

(ii) We observe that $\boldsymbol{L}^p_\sigma(\Omega) = \boldsymbol{L}^p(\Omega)/\boldsymbol{G}^{\,p}(\Omega)$ and $(\boldsymbol{L}^p_\sigma(\Omega))' = \boldsymbol{G}^{\,p}(\Omega)^\perp$. As $\boldsymbol{L}^{p'}_\sigma(\Omega)$ is a closed subspace of $\boldsymbol{L}^{p'}(\Omega)$. Hence, if we prove that $\boldsymbol{L}^{p'}_\sigma(\Omega)^\perp = \boldsymbol{G}^{\,p}(\Omega)$ this will imply

$$\boldsymbol{G}^{\,p}(\Omega)^\perp = (\boldsymbol{L}^{p'}_\sigma(\Omega)^\perp)^\perp = \overline{\boldsymbol{L}^{p'}_\sigma(\Omega)} = \boldsymbol{L}^{p'}_\sigma(\Omega),$$

which is the required result because $\boldsymbol{G}^{\,p}(\Omega)^\perp = (\boldsymbol{L}^p_\sigma(\Omega))'$. First, let $\boldsymbol{u} \in \boldsymbol{G}^{\,p}(\Omega)$. Then, we have for any $\boldsymbol{v} \in \boldsymbol{L}^{p'}_\sigma(\Omega)$

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\,\pi\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = 0,$$

because π belongs to $W_0^{1,p}(\Omega)$. Hence $\boldsymbol{u} \in \boldsymbol{L}^{p'}_\sigma(\Omega)^\perp$ and $\boldsymbol{G}^{\,p}(\Omega) \subset \boldsymbol{L}^{p'}_\sigma(\Omega)^\perp$. Conversely, let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ such that for any $\boldsymbol{v} \in \boldsymbol{L}^{p'}_\sigma(\Omega)$:

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = 0. \tag{5.32}$$

By choosing $\boldsymbol{v}$ in the space $\mathcal{V} = \{\boldsymbol{v} \in \mathcal{D}(\Omega),\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega\}$ and using De Rham's Lemma, we deduce that $\boldsymbol{u} = \nabla\,\pi$, where $\pi \in W^{\,1,p}(\Omega)$. As π is unique up to an additive constant, we can choose this constant in such a way that $\int_\Gamma \pi\,\mathrm{d}\boldsymbol{s} = 0$. From (5.32), we obtain

$$\forall\,\boldsymbol{v} \in \boldsymbol{L}^{p'}_\sigma(\Omega), \quad \langle \pi,\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_\Gamma = 0, \tag{5.33}$$

where $\langle\cdot,\,\cdot\rangle_\Gamma$ denotes the duality bracket $W^{\,1-1/p,p}(\Gamma)\times W^{\,-1/p',p'}(\Gamma)$. Let now $\mu \in W^{\,-1/p',p'}(\Gamma)$ and $\theta \in W^{\,1,p'}(\Omega)$ a solution of the following Neumann problem

$$\Delta\,\theta = 0 \ \text{ in } \Omega, \qquad \frac{\partial\,\theta}{\partial\,\boldsymbol{n}} = \mu - \frac{1}{|\Gamma|}\langle \mu,\, 1\rangle_\Gamma \ \text{ on } \Gamma. \tag{5.34}$$

Next, we set $\boldsymbol{v} = \nabla\,\theta$. We deduce from (5.33) and (5.34) that:

$$\langle \pi,\, \mu - \frac{1}{|\Gamma|}\langle \mu,\, 1\rangle_\Gamma\rangle_\Gamma = 0.$$

Then

$$\forall \mu \in W^{\,-1/p',p'}(\Gamma), \quad \langle \pi,\, \mu\rangle_\Gamma = 0,$$

which implies that $\pi = 0$ on Γ. Consequently, $\boldsymbol{u}$ belongs to $\boldsymbol{G}^{\,p}(\Omega)$. Therefore, $\boldsymbol{L}^{p'}_\sigma(\Omega)^\perp \subset \boldsymbol{G}^{\,p}(\Omega)$ which finishes the proof. □

Now, we introduce the space:

$$\boldsymbol{M}^{\,p}(\Omega) = \big\{(\boldsymbol{v},\, \pi) \in \boldsymbol{L}^p_\sigma(\Omega) \times L^p(\Omega);\ \ -\Delta\,\boldsymbol{v} + \nabla\,\pi \in [\boldsymbol{H}^{\,p'}_0(\mathbf{curl},\, \Omega)]'\big\},$$

which is a Banach space for the norm:

$$\|(\boldsymbol{v},\, \pi)\|_{\boldsymbol{M}^{\,p}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{L^p(\Omega)} + \|-\Delta\,\boldsymbol{v} + \nabla\,\pi\|_{[\boldsymbol{H}^{\,p'}_0(\mathbf{curl},\,\Omega)]'}.$$

Lemma 5.5. *The space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{M}^p(\Omega)$.*

Proof. Let $\boldsymbol{\ell}$ in $[\boldsymbol{M}^{\,p}(\Omega)]'$ such that:

$$\forall\,(\boldsymbol{v},\, \pi) \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega}), \quad \langle \boldsymbol{\ell},\, (\boldsymbol{v},\, \pi)\rangle = 0. \tag{5.35}$$

There exist $\boldsymbol{f} \in \boldsymbol{L}^{p'}_\sigma(\Omega)$, $\lambda \in L^{p'}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{H}^{\,p'}_0(\mathbf{curl},\, \Omega)$, such that for any $(\boldsymbol{v}, \pi) \in \boldsymbol{M}^{\,p}(\Omega)$,

$$\begin{aligned}\langle \boldsymbol{\ell}, (\boldsymbol{v},\, \pi)\rangle = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \lambda\,\pi\,\mathrm{d}\boldsymbol{x}\\ + \langle -\Delta\,\boldsymbol{v} + \nabla\,\pi,\, \boldsymbol{g}\rangle_{[\boldsymbol{H}^{\,p'}_0(\mathbf{curl},\,\Omega)]'\times\boldsymbol{H}^{\,p'}_0(\mathbf{curl},\,\Omega)}\end{aligned} \tag{5.36}$$

where we have used that $[\boldsymbol{L}^p_\sigma(\Omega)]' = \boldsymbol{L}^{p'}_\sigma(\Omega)$ (as in Lemma 5.4). In particular, if $(\boldsymbol{v},\, \pi) \in \boldsymbol{\mathcal{D}}_\sigma(\Omega) \times \mathcal{D}(\Omega)$, we have

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \lambda\,\pi\,\mathrm{d}\boldsymbol{x} + \langle -\Delta\,\boldsymbol{g},\, \boldsymbol{v}\rangle_\Omega - \langle \mathrm{div}\,\boldsymbol{g},\, \pi\rangle_\Omega = 0,$$

where $\langle\cdot,\,\cdot\rangle_\Omega$ is the duality bracket between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. Particularly, if $\pi = 0$, we obtain for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\Omega)$:

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \langle -\Delta\,\boldsymbol{g},\, \boldsymbol{v}\rangle_\Omega = 0.$$

Since $\boldsymbol{f} - \Delta\, \boldsymbol{g}$ belongs to $\boldsymbol{W}^{-2,p}(\Omega)$, using De Rham's Lemma (see Ref. 4), there exists $\theta \in W^{-1,p}(\Omega)$, unique up to an additive constant, such that

$$\boldsymbol{f} - \Delta\, \boldsymbol{g} = \nabla\, \theta \ \text{ in}\, \Omega.$$

If we choose now $\boldsymbol{v} = \boldsymbol{0}$, we obtain for any $\pi \in \mathcal{D}(\Omega)$:

$$\int_\Omega \lambda\, \pi\, \mathrm{d}\boldsymbol{x} - \langle \operatorname{div} \boldsymbol{g},\, \pi \rangle_\Omega = 0,$$

which implies that $\lambda = \operatorname{div} \boldsymbol{g}$ in Ω. Observe that we can extend by zero the functions $\boldsymbol{f}$, λ and $\boldsymbol{g}$ in such a way that

$$\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^{p'}(\mathbb{R}^3),\ \widetilde{\lambda} \in L^p(\mathbb{R}^3) \quad \text{and} \quad \widetilde{\boldsymbol{g}} \in \boldsymbol{H}^{p'}(\mathbf{curl},\, \mathbb{R}^3).$$

Moreover, for any $\chi \in \mathcal{D}(\mathbb{R}^3)$ such that $\Delta\, \chi = 0$ in Ω, we have by (5.36) with $\boldsymbol{v} = \nabla\, \chi|_\Omega$:

$$\int_\Omega \boldsymbol{f} \cdot \nabla\, \chi\, \mathrm{d}\boldsymbol{x} = 0.$$

Let $\mu \in W^{\frac{1}{p'},p}(\Gamma)$. By the density of $\mathcal{D}(\Gamma)$ in $W^{\frac{1}{p'},p}(\Gamma)$, there exists a sequence $\mu_k \in \mathcal{D}(\Gamma)$ such that μ_k converges to μ in $W^{\frac{1}{p'},p}(\Gamma)$. Let now φ_k be the solution of the problem

$$-\Delta\, \varphi_k = 0 \ \text{ in } \Omega \quad \text{and} \quad \varphi_k = \mu_k \ \text{ on } \Gamma.$$

We know that φ_k belongs to $\mathcal{C}^\infty(\Omega)$. Let $\psi_k \in \mathcal{D}(\mathbb{R}^3)$ an extension of φ_k to $\mathbb{R}^3$. Then φ_k belongs to $\mathcal{D}(\overline{\Omega})$ and we have,

$$0 = \int_\Omega \boldsymbol{f} \cdot \nabla\, \varphi_k = \langle \boldsymbol{f} \cdot \boldsymbol{n},\, \mu_k \rangle_{W^{-\frac{1}{p'},p'}(\Gamma) \times W^{\frac{1}{p'},p}(\Gamma)}.$$

So, $\langle \boldsymbol{f} \cdot \boldsymbol{n},\, \mu \rangle_{W^{-\frac{1}{p'},p'}(\Gamma) \times W^{\frac{1}{p'},p}(\Gamma)} = 0$ for any $\mu \in W^{\frac{1}{p'},p}(\Gamma)$.

Consequently, $\boldsymbol{f} \cdot \boldsymbol{n} = 0$ on Γ and $\operatorname{div} \widetilde{\boldsymbol{f}} = 0$ in $\mathbb{R}^3$. Now, we take $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ with $\operatorname{div} \boldsymbol{\varphi} = 0$ and $q \in \mathcal{D}(\mathbb{R}^3)$. We obtain by (5.35) and (5.36):

$$\int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}} \cdot \boldsymbol{\varphi}\, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3} \widetilde{\lambda}\, q\, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3} (-\Delta\, \boldsymbol{\varphi} + \nabla\, q) \cdot \widetilde{\boldsymbol{g}}\, \mathrm{d}\boldsymbol{x} = 0. \tag{5.37}$$

In particular, if $q = 0$, then by De Rham's Lemma:

$$\widetilde{\boldsymbol{f}} - \Delta\, \widetilde{\boldsymbol{g}} = \nabla\, \theta_0 \quad \text{in } \mathbb{R}^3, \tag{5.38}$$

with $\theta_0 \in \mathcal{D}'(\mathbb{R}^3)$. Since $\operatorname{div} \widetilde{\boldsymbol{f}} = 0$ in $\mathbb{R}^3$, then $-\Delta \operatorname{div} \widetilde{\boldsymbol{g}} = \Delta\, \theta_0$ in $\mathbb{R}^3$. But $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^{p'}(\mathbb{R}^3)$ and $\operatorname{supp} \widetilde{\boldsymbol{f}}$ is compact, then $\widetilde{\boldsymbol{f}} \in \boldsymbol{W}_0^{-2,p'}(\mathbb{R}^3)$ where $\boldsymbol{W}_0^{-2,p'}(\mathbb{R}^3)$ is the dual space of the weighted sobolev space

$$\boldsymbol{W}_0^{2,p}(\mathbb{R}^3) = \{\boldsymbol{v} \in \boldsymbol{D}'(\mathbb{R}^3),\ \frac{\boldsymbol{v}}{\omega_0} \in \boldsymbol{L}^p(\mathbb{R}^3),\ \frac{\nabla\, \boldsymbol{v}}{\omega_1} \in \boldsymbol{L}^p(\mathbb{R}^3),\ D^2 \boldsymbol{v} \in \boldsymbol{L}^p(\mathbb{R}^3)\},$$

with $\omega_0 = (1+|\boldsymbol{x}|)^2$ if $p \notin \{3/2, 3\}$, $\omega_0 = (1+|\boldsymbol{x}|)^2 \ln(2+|\boldsymbol{x}|)$ if $p \in \{3/2, 3\}$,

$\omega_1 = (1+|\boldsymbol{x}|)$ if $p \neq 3$, $\omega_1 = (1+|\boldsymbol{x}|)\ln(2+|\boldsymbol{x}|)$ if $p = 3$.

Consequently $\nabla\,\theta_0 \in \boldsymbol{W}_0^{-2,p'}(\mathbb{R}^3)$. We deduce that $\theta_0 \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$ and then $\theta_0 = -\mathrm{div}\,\widetilde{\boldsymbol{g}}$ in $\mathbb{R}^3$. By taking in (5.38) the restriction to Ω, we obtain $\boldsymbol{f} - \Delta\,\boldsymbol{g} = \nabla\,\theta_0|_\Omega$. As Ω is connected, there exists a unique constant a such that $\theta = \theta_0 + a$ in Ω. Hence, relation (5.38) becomes $\Delta\,\widetilde{\boldsymbol{g}} - \nabla\,\mathrm{div}\,\widetilde{\boldsymbol{g}} = \widetilde{\boldsymbol{f}}$ in $\mathbb{R}^3$. That means that $\mathbf{curl}\,\widetilde{\boldsymbol{g}} \in \boldsymbol{H}^{\,p'}(\mathbf{curl},\,\mathbb{R}^3)$ and then $\mathbf{curl}\,\boldsymbol{g} \in \boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)$.

Moreover, taking $\boldsymbol{\varphi} = \mathbf{0}$, we obtain from (5.37) that $\widetilde{\lambda} = \mathrm{div}\,\widetilde{\boldsymbol{g}}$ in $\mathbb{R}^3$. We deduce that $\mathrm{div}\,\widetilde{\boldsymbol{g}}$ belongs to $L^{p'}(\mathbb{R}^3)$ and to $L^{p'}(\Omega)$ by restriction on Ω. Then, $\boldsymbol{g} \in \boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega)$. As $\boldsymbol{g} \in \boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)$, then $\boldsymbol{g} \in \boldsymbol{W}_0^{1,p'}(\Omega)$. Moreover, as $\mathbf{curl}\,\boldsymbol{g} \in \boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)$, then $\boldsymbol{g}$ belongs to $\boldsymbol{N}^{\,p'}(\Omega)$ and there exists a sequence $(\boldsymbol{g}_k)_k \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\boldsymbol{g}_k$ converges to $\boldsymbol{g}$ in $\boldsymbol{N}^{\,p'}(\Omega)$ when $k \to \infty$. Finally, we consider $(\boldsymbol{v},\,\pi) \in \boldsymbol{M}^{\,p}(\Omega)$. Observe that

$$
\begin{aligned}
\langle \boldsymbol{\ell}, (\boldsymbol{v},\,\pi)\rangle &= -\int_\Omega \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{g}\,\mathrm{d}\boldsymbol{x} + \langle -\Delta\,\boldsymbol{v} + \nabla\,\pi,\ \boldsymbol{g}\rangle_\Omega \\
&= \lim_{k\to\infty}\Big(-\int_\Omega \mathbf{curl}\,\boldsymbol{g}_k\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{g}_k\,\mathrm{d}\boldsymbol{x} + \\
&\quad + \int_\Omega \mathbf{curl}\,\boldsymbol{g}_k\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{g}_k\,\mathrm{d}\boldsymbol{x}\Big) = 0,
\end{aligned}
$$

where $\langle\,\cdot,\,\cdot\,\rangle_\Omega$ denotes the duality bracket $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)$. Therefore, $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{M}^{\,p}(\Omega)$. □

In order to give meaning to the trace of a very weak solution of the Stokes problem $(\mathcal{S}_N)$, we need to introduce the space:

$$\boldsymbol{T}_N^{\,p}(\Omega) = \big\{\boldsymbol{\varphi} \in \boldsymbol{W}^{\,2,p}(\Omega);\ \boldsymbol{\varphi}\times\boldsymbol{n} = \mathbf{0}\ \text{ and }\ \mathrm{div}\,\boldsymbol{\varphi} = \mathbf{0}\ \text{ on}\,\Gamma\big\}.$$

Theorem 5.4. *The linear mapping $\gamma\,:\,(\boldsymbol{u},\,\pi) \mapsto (\boldsymbol{u}\times\boldsymbol{n},\,\pi|_\Gamma)$ defined on $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by γ, from $\boldsymbol{M}^p(\Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma) \times W^{\,-1/p,p}(\Gamma)$, and we have the Green formula: for any $(\boldsymbol{u},\,\pi) \in \boldsymbol{M}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{T}_N^{p'}(\Omega)$,*

$$
\begin{aligned}
\langle -\Delta\,\boldsymbol{u} + \nabla\,\pi,\ \boldsymbol{\varphi}\rangle_\Omega = -\int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi} + \langle \boldsymbol{u}\times\boldsymbol{n},\,\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma - \int_\Omega \pi\,\mathrm{div}\,\boldsymbol{\varphi} \\
+ \langle \pi,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma
\end{aligned}
\tag{5.39}
$$

where $\langle\,\cdot,\,\cdot\,\rangle_\Omega$ denotes the duality bracket $[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)$ and $\langle\,\cdot,\,\cdot\,\rangle_\Gamma$ denotes the duality bracket $W^{\,-1/p,p}(\Gamma) \times W^{\,1/p,p'}(\Gamma)$ or $\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{1/p,p'}(\Gamma)$.

Proof. Let $(\boldsymbol{u},\pi) \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$, then formula (5.39) is valid for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{\,p'}(\Omega)$. Let $\boldsymbol{\mu} \in \boldsymbol{W}^{1/p,p'}(\Gamma)$. Then, there exists a function $\boldsymbol{\varphi} \in \boldsymbol{W}^{\,2,p'}(\Omega)$ such that:

$$\boldsymbol{\varphi} = \mathbf{0} \quad \text{and} \quad \frac{\partial\, \boldsymbol{\varphi}}{\partial\, \boldsymbol{n}} = \boldsymbol{\mu} \times \boldsymbol{n} \quad \text{on } \Gamma$$

and verifying:

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)} \leq C \|\boldsymbol{\mu}\|_{\boldsymbol{W}^{\,1/p,p'}(\Gamma)}. \tag{5.40}$$

Moreover, since $\boldsymbol{\varphi} = \mathbf{0}$ on Γ, using (5.10) we obtain $\mathbf{curl}\, \boldsymbol{\varphi} = -\boldsymbol{\mu}_t$ on Γ. Using (5.8), the function $\boldsymbol{\varphi}$ belongs to $\boldsymbol{T}_N^{p'}(\Omega)$. Consequently,

$$\begin{aligned}
\Big| \langle \boldsymbol{u} \times \boldsymbol{n},\, \boldsymbol{\mu} \rangle_\Gamma \Big| &= \Big| \langle \boldsymbol{u} \times \boldsymbol{n},\, \mathbf{curl}\, \boldsymbol{\varphi} \rangle_\Gamma \Big| \\
&\leq \| -\Delta\, \boldsymbol{u} + \nabla\, \pi \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'} \|\boldsymbol{\varphi}\|_{\boldsymbol{H}^{\,p'}(\mathbf{curl},\, \Omega)} + \\
&\quad + \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \|\Delta\, \boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\pi\|_{L^{\,p}(\Omega)} \|\operatorname{div} \boldsymbol{\varphi}\|_{L^{\,p'}(\Omega)} \\
&\leq C \|(\boldsymbol{u},\, \pi)\|_{\boldsymbol{M}^{\,p}(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)}.
\end{aligned}$$

Thus, using (5.40), we obtain for any $(\boldsymbol{u},\, \pi) \in \boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$:

$$\|\boldsymbol{u} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \leq C \|(\boldsymbol{u}, \pi)\|_{\boldsymbol{M}^{\,p}(\Omega)}.$$

Concerning the trace of π, the same reasoning leads only to show that this trace belongs to $W^{\,-1-1/p,p}(\Gamma)$. But, we have $\Delta\, \pi \in W^{\,-1,p}(\Omega)$ and $\pi \in L^p(\Omega)$. Then due to Ref. 5, the trace of π on Γ belongs to $W^{\,-1/p,p}(\Gamma)$. Moreover, we have:

$$\|\pi\|_{W^{\,-1/p,p}(\Gamma)} \leq C \Big(\|\pi\|_{L^p(\Omega)} + \|\Delta\, \pi\|_{W^{\,-1,p}(\Omega)} \Big).$$

But $-\Delta\, \boldsymbol{u} + \nabla \pi \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'$, there exist $\boldsymbol{\psi}_0 \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{L}^p(\Omega)$ such that $-\Delta\, \boldsymbol{u} + \nabla \pi = \boldsymbol{\psi}_0 + \mathbf{curl}\, \boldsymbol{h}$ with

$$\|\boldsymbol{\psi}_0\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h}\|_{\boldsymbol{L}^p(\Omega)} \leq C \| -\Delta\, \boldsymbol{u} + \nabla \pi \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'}.$$

Then, by taking the divergence we obtain

$$\|\Delta\, \pi\|_{W^{\,-1,p}(\Omega)} = \|\operatorname{div} \boldsymbol{\psi}_0\|_{W^{\,-1,p}(\Omega)} \leq C \|\boldsymbol{\psi}_0\|_{\boldsymbol{L}^p(\Omega)} \leq C \| -\Delta\, \boldsymbol{u} + \nabla \pi \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'}.$$

As a consequence, we have

$$\begin{aligned}
\|\pi\|_{W^{\,-1/p,p}(\Gamma)} &\leq C \Big(\|\pi\|_{L^p(\Omega)} + \| -\Delta\, \boldsymbol{u} + \nabla \pi \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'} \Big) \\
&\leq C \|(\boldsymbol{u},\, \pi)\|_{\boldsymbol{M}^p(\Omega)}.
\end{aligned}$$

Therefore, we obtain that the linear mapping $\gamma :\ (\boldsymbol{u},\, \pi) \mapsto (\boldsymbol{u}_\Gamma \times \boldsymbol{n},\, \pi|_\Gamma)$ defined on the space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{M}^{\,p}(\Omega)$. Since $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{M}^{\,p}(\Omega)$, then we can extend this mapping from $\boldsymbol{M}^{\,p}(\Omega)$ into $\boldsymbol{W}^{-1/p,p}(\Gamma) \times W^{\,-1/p,p}(\Gamma)$ and the Green formula (5.39) holds for any $(\boldsymbol{u},\, \pi) \in \boldsymbol{M}^{\,p}(\Omega)$ and for any $\boldsymbol{\varphi} \in \boldsymbol{T}_N^{p'}(\Omega)$. $\square$

Theorem 5.5. (**Very weak solutions for** $(\mathcal{S}_{\rm N})$) *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 with:*

$$\boldsymbol{f}\in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]',\;\; \boldsymbol{g}\times\boldsymbol{n}\in\,\boldsymbol{W}^{-1/p,p}(\Gamma),\;\;\pi_0\in W^{\,-1/p,p}(\Gamma),$$

and satisfying the compatibility condition (5.20) *where we replace the integral by a bracket duality. Then, the Stokes problem $(\mathcal{S}_N)$ has exactly one solution $\boldsymbol{u}\in\boldsymbol{L}^p(\Omega)$ and $\pi\in L^p(\Omega)$. Moreover, there exists a constant $C>0$ depending only on p and Ω such that:*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}+\|\pi\|_{L^p(\Omega)}\le C\Big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}+\|\,\boldsymbol{g}\,\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}+ \\ +\|\pi_0\|_{W^{\,-1/p,p}(\Gamma)}\Big). \qquad (5.41)$$

Proof.

(i) **First step:** Thanks to the Green formula (5.39), it is easy to verify that $(\boldsymbol{u},\pi)\in\boldsymbol{L}^p(\Omega)\times L^p(\Omega)$ is solution of problem $(\mathcal{S}_N)$, without the last flux condition, is equivalent to the variational formulation: Find $(\boldsymbol{u},\,\pi)\in\boldsymbol{L}^p(\Omega)\times L^p(\Omega)$ such that for any $\boldsymbol{\varphi}\in\boldsymbol{T}_N^{\,p'}(\Omega)$ and $q\in W_0^{1,p'}(\Omega)$,

$$\int_\Omega\boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\int_\Omega\pi\,\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}=-\,\langle\,\boldsymbol{f},\,\boldsymbol{\varphi}\,\rangle_\Omega+\langle\,\boldsymbol{g}\times\boldsymbol{n},\mathbf{curl}\,\boldsymbol{\varphi}\,\rangle_\Gamma+ \\ +\,\langle\pi_0,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma, \qquad (5.42)$$
$$\int_\Omega\boldsymbol{u}\cdot\nabla\,q\,\mathrm{d}\boldsymbol{x}=0.$$

Indeed, let $(\boldsymbol{u},\,\pi)\in\boldsymbol{L}^p(\Omega)\times L^p(\Omega)$ be a solution to (5.42). It is clear that:

$$-\Delta\,\boldsymbol{u}+\nabla\,\pi=\boldsymbol{f}\quad\text{ and }\quad\mathrm{div}\,\boldsymbol{u}=0\quad\text{in }\Omega.$$

Using Green formula (5.39), we obtain for any $\boldsymbol{\varphi}\in\boldsymbol{T}_N^{p'}(\Omega)$:

$$-\int_\Omega\boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\langle\boldsymbol{u}\times\boldsymbol{n},\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma-\int_\Omega\pi\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}+\langle\pi_0,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma=\langle\boldsymbol{f},\,\boldsymbol{\varphi}\rangle_\Omega.$$

Then, we deduce that for any $\boldsymbol{\varphi}\in\boldsymbol{T}_N^{\,p'}(\Omega)$,

$$\langle\,\boldsymbol{u}\times\boldsymbol{n},\,\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma+\langle\pi,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma=\langle\boldsymbol{g}\times\boldsymbol{n},\,\mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma+\langle\pi_0,\,\boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma.$$

Let $\boldsymbol{\mu}\in\,\boldsymbol{W}^{1/p,p'}(\Gamma)$. Then, there exists a function $\boldsymbol{\varphi}\in\,\boldsymbol{W}^{\,2,p'}(\Omega)$ such that:

$$\boldsymbol{\varphi}=\mathbf{0}\;\;\text{and}\;\;\frac{\partial\,\boldsymbol{\varphi}}{\partial\,\boldsymbol{n}}=\boldsymbol{\mu}_{\mathbf{t}}\quad\text{on }\Gamma,$$

and this implies that $\mathbf{curl}\,\boldsymbol{\varphi}\times\boldsymbol{n}=-\boldsymbol{\mu}_t$ and $\mathrm{div}\,\boldsymbol{\varphi}=0$ on Γ, that means that $\boldsymbol{\varphi}\in\boldsymbol{T}_N^{\,p'}(\Omega)$. We deduce that for all $\boldsymbol{\mu}\in\,\boldsymbol{W}^{1/p,p'}(\Gamma)$,

$$\langle\, \boldsymbol{u}\times\boldsymbol{n},\, \boldsymbol{\mu}\rangle_\Gamma = \langle \boldsymbol{g}\times\boldsymbol{n},\, \boldsymbol{\mu}\rangle_\Gamma.$$

Consequently $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ on Γ. Let us prove now that $\pi = \pi_0$ on Γ. For any $\lambda \in W^{\,1+1/p,p'}(\Gamma)$, there exists a function $\boldsymbol{\varphi} \in \boldsymbol{W}^{\,2,p'}(\Omega)$ such that: $\boldsymbol{\varphi} = \lambda\boldsymbol{n}$ and $\dfrac{\partial\boldsymbol{\varphi}}{\partial\boldsymbol{n}}\cdot\boldsymbol{n} = -K\lambda$ on Γ, where K is the curvature of Γ. Observe that $\boldsymbol{\varphi}$ belongs to $\boldsymbol{T}_N^{p'}(\Omega)$ and then for any $\lambda\in W^{\,1+1/p,p'}(\Gamma)$ we have:

$$\langle \pi,\, \lambda\rangle_{W^{\,-1-1/p,p}(\Gamma)\times W^{\,1+1/p,p'}(\Gamma)} = \langle \pi_0,\, \lambda\rangle_{W^{\,-1-1/p,p}(\Gamma)\times W^{\,1+1/p,p'}(\Gamma)},$$

and then, $\pi = \pi_0$ on Γ.

The converse is a simple consequence of the Green formula (5.39) and the fact that for any $\boldsymbol{\varphi}\in \boldsymbol{T}_N^{p'}(\Omega)$:

$$\int_\Omega \boldsymbol{u}\cdot\nabla\,\mathrm{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{u}\cdot\boldsymbol{n},\, \mathrm{div}\,\boldsymbol{\varphi}\rangle_\Omega = 0.$$

(ii) **Second step** : Let's now solve problem (5.42). We know due to Corollary 5.6 that for any $(\boldsymbol{F},\,\chi)\in \boldsymbol{L}^{p'}(\Omega)\perp \boldsymbol{K}_N^{\,p}(\Omega)\times W_0^{1,p'}(\Omega)$, there exists a unique $\boldsymbol{\varphi}\in \boldsymbol{W}^{\,2,p'}(\Omega)$ and $q\in W_0^{1,p'}(\Omega)$ satisfying :

$$\begin{cases} -\Delta\,\boldsymbol{\varphi} + \nabla\, q = \boldsymbol{F} \quad \text{and} \qquad \mathrm{div}\,\boldsymbol{\varphi} = \chi & \text{in}\,\Omega,\\ \boldsymbol{\varphi}\times\boldsymbol{n} = \boldsymbol{0} \ \ \text{and} \ \ q = 0 & \text{on}\,\Gamma,\\ \langle\boldsymbol{\varphi}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0 \ \text{ for any } 1\le i\le I, & \end{cases}$$

with the estimate

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{\,2,p'}(\Omega)} + \|q\|_{W^{\,1,p'}(\Omega)} \le C\big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\chi\|_{W_0^{1,p'}(\Omega)}\big).$$

From this bound, we have

$$\begin{aligned} &\Big|\quad \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g}\times\boldsymbol{n},\, \mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma - \langle\pi_0,\, \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma\Big|\\ &\le C\,\Big(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{\,-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{\,-1/p,p}(\Gamma)}\Big)\times \\ &\times\quad \Big(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\chi\|_{W_0^{1,p'}(\Omega)}\Big). \end{aligned} \tag{5.43}$$

In other words, we can say that the linear mapping:

$$(\boldsymbol{F},\,\chi)\mapsto \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g}\times\boldsymbol{n},\, \mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma - \langle\pi_0,\, \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma$$

defines an element of the dual space of $\big(\boldsymbol{L}^{p'}(\Omega)\perp\boldsymbol{K}_N^{\,p}(\Omega)\big)\times W_0^{1,p'}(\Omega)$, that means that there exists a unique $(\boldsymbol{u},\,\pi)\in\big(\boldsymbol{L}^p(\Omega)/\boldsymbol{K}_N^{\,p}(\Omega)\big)\times W^{\,-1,p}(\Omega)$ satisfying

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{F}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \pi\chi\,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f},\, \boldsymbol{\varphi}\rangle_\Omega - \langle \boldsymbol{g}\times\boldsymbol{n},\, \mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma - \langle\pi_0,\, \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma.$$

A such solution $(\boldsymbol{u},\, \pi)$ satisfies the problem $(\mathcal{S}_N)$ without the last condition but we have only to set

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \sum_{i=1}^{I} \langle \boldsymbol{u}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i}\, \nabla\, q_i^N.$$

It is clear that $(\widetilde{\boldsymbol{u}},\, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)$ is also solution of $(\mathcal{S}_N)$ and satisfies its last condition. Moreover, $\pi \in W^{-1,p}(\Omega)$ satisfies:

$$\Delta\,\pi = \operatorname{div}\boldsymbol{f} \ \text{ in}\,\Omega \quad \text{and} \quad \pi = \pi_0 \ \text{ on}\,\Gamma.$$

Since $\operatorname{div}\boldsymbol{f} \in W^{-1,p}(\Omega)$ and $\pi_0 \in W^{-1/p,p}(\Gamma)$, we deduce from Ref. 5 that π belongs to $L^p(\Omega)$. Finally, the estimate (5.41) can be deduced from (5.43). $\square$

Concerning now the existence of very weak solutions for the problem (E_N) ,we need to introduce the following space:

$$\boldsymbol{\mathcal{M}}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p_\sigma(\Omega);\ \ \Delta\,\boldsymbol{v} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'\},$$

which is a Banach space for the norm:

$$\|\boldsymbol{v}\|_{\boldsymbol{\mathcal{M}}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta\,\boldsymbol{v}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'}.$$

Using the same arguments given in the Lemma 5.5, we can prove that the space $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ is dense in $\boldsymbol{\mathcal{M}}^p(\Omega)$. To give a sense to the trace of functions which belong to $\boldsymbol{\mathcal{M}}^p(\Omega)$, we have the following lemma, where the proof is very similar to that of Theorem 5.4.

Lemma 5.6. *The linear mapping $\gamma\ :\ \boldsymbol{u} \longmapsto \boldsymbol{u}\times\boldsymbol{n}|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$\gamma\ :\ \boldsymbol{\mathcal{M}}^p(\Omega) \longmapsto \boldsymbol{W}^{-1/p,p}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{u} \in \boldsymbol{\mathcal{M}}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{T}_N^{p'}(\Omega)$,

$$\langle \Delta\,\boldsymbol{u},\, \boldsymbol{\varphi}\rangle_\Omega = \int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{u}\times\boldsymbol{n},\, \mathbf{curl}\,\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times \boldsymbol{W}^{1/p,p'}(\Gamma)},$$

where the duality on Ω is the following

$$\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'\times \boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)}.$$

As a consequence, as for Theorem 5.5, we have the following result concerning the very weak solutions for the elliptic problem (E_N).

Corollary 5.7. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$ with $\operatorname{div}\boldsymbol{f} = 0$ in Ω satisfying the compatibility condition* (5.2) *and let $\boldsymbol{g} \in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Then the problem (E_N) has a unique solution $\boldsymbol{\xi} \in \boldsymbol{L}^p(\Omega)$, with the estimate*

$$\|\,\boldsymbol{\xi}\,\|_{\boldsymbol{L}^p(\Omega)} \leq C\big(\|\,\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

5.4. *A variant of the system* $(\mathcal{S}_N)$

As it is shown in the previous sections, in order to solve problem $(\mathcal{S}_N)$, the data must satisfy the compatibility condition (5.20). Now, what happen if this condition is not satisfied? As will appear, the answer strongly depends on the following variant of the Stokes problem $(\mathcal{S}_N)$: Find functions $\boldsymbol{u}$, π and constants c_i for $i = 1, \ldots, I$, such that:

$$(\mathcal{S}'_N) \quad \begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on}\, \Gamma, \\ \pi = \pi_0 \ \text{ on } \Gamma_0 \quad \text{and} \quad \pi = \pi_0 + c_i & \text{on}\, \Gamma_i, \ \ 1 \leq i \leq I \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \ \ 1 \leq i \leq I, \end{cases}$$

situation that we can be found in Ref. 14. Let us compare with our approach.

Theorem 5.6. *Let* $\boldsymbol{f}$, $\boldsymbol{g}$ *and* π_0 *such that:*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]', \quad \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \quad \pi_0 \in W^{\,1-1/p,p}(\Gamma).$$

Then, the problem $(\mathcal{S}'_N)$ *has a unique solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$, $\pi \in W^{\,1,p}(\Omega)$ *and constants* $c_1, \ldots, c_I$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{W^{\,1,p}(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \|\pi_0\|_{W^{\,1-1/p,p}(\Gamma)}\big),$$

and where $c_1, \ldots, c_I$ *are given by* (5.44). *In particular, if* $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, *then* $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.

Proof.

(i) We suppose that $\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'$, $\boldsymbol{g} \in \boldsymbol{W}^{\,1-1/p,p}(\Gamma)$ and $\pi_0 \in W^{\,1-1/p,p}(\Gamma)$. Observe that the following problem

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = 0 & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on}\, \Gamma, \\ \pi = \pi_0 & \text{on}\, \Gamma_0, \\ \pi = \pi_0 + \langle \boldsymbol{f},\, \nabla\, q_i^N \rangle_\Omega - \langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n} \rangle_\Gamma & \text{on}\, \Gamma_i, \ \ 1 \leq i \leq I, \\ \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \end{cases}$$

has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{\,1,p}(\Omega) \times W^{\,1,p}(\Omega)$ since the compatibility condition (5.20) is verified. The brackets on Ω denote the duality between

$[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'$ and $\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)$ and the brackets on Γ denote the duality between $W^{1-1/p,p}(\Gamma)$ and $W^{-1/p',p'}(\Gamma)$. For $i=1,\ldots,I$, we set $\boldsymbol{c}=(c_1,\ldots,c_I)$ where

$$c_i = \langle \boldsymbol{f},\,\nabla q_i^N\rangle_\Omega - \langle \pi_0,\,\nabla q_i^N\cdot\boldsymbol{n}\rangle_\Gamma. \tag{5.44}$$

Finally, $(\boldsymbol{u},\,\pi,\,\boldsymbol{c})\in \boldsymbol{W}^{1,p}(\Omega)\times W^{1,p}(\Omega)\times\mathbb{R}^I$ is the solution of $(\mathcal{S}'_N)$.

(ii) Let $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$ the solution of problem $(\mathcal{S}'_N)$ obtained by the previous point. We suppose now that $\boldsymbol{f}\in\boldsymbol{L}^p(\Omega)$, $\boldsymbol{g}\in\boldsymbol{W}^{2-1/p,p}(\Gamma)$ and we set $\boldsymbol{z}=\mathbf{curl}\,\boldsymbol{u}$. Since $\boldsymbol{u}\times\boldsymbol{n}\in\boldsymbol{W}^{2-1/p,p}(\Gamma)$, by Lemma 5.3, we have $\boldsymbol{z}\cdot\boldsymbol{n}\in W^{1-1/p,p}(\Gamma)$. By Theorem 3.5, the function $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. Then, $\boldsymbol{u}$ satisfies

$$\boldsymbol{u}\in\boldsymbol{L}^p(\Omega),\ \ \mathrm{div}\,\boldsymbol{u}=0,\ \ \mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)\ \text{ and }\ \boldsymbol{u}\times\boldsymbol{n}\in W^{2-1/p,p}(\Gamma).$$

We deduce from Corollary 3.5 that $\boldsymbol{u}\in\boldsymbol{W}^{2,p}(\Omega)$. □

Remark 5.4. Observe that if we suppose that the compatibility condition (5.20) is verified, we have that $c_i=0$ for all $i=1,\ldots,I$. Then, we have reduced to solve the problem $(\mathcal{S}'_N)$ without the constant c_i and $(\mathcal{S}'_N)$ is anything other then $(\mathcal{S}_N)$.

The assumption on $\boldsymbol{f}$ in Theorem 5.6 or Theorem 5.2 can be weakened by considering the space defined for all $1<r,\,p<\infty$:

$$\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)=\{\boldsymbol{\varphi}\in\boldsymbol{L}^r(\Omega);\ \ \mathbf{curl}\,\boldsymbol{\varphi}\in\boldsymbol{L}^p(\Omega),\ \ \boldsymbol{\varphi}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma\},$$

which is a Banach space for the norm

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{H}_0^{r,\,p}(\mathbf{curl},\,\Omega)}=\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^r(\Omega)}+\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(\Omega)}.$$

We can prove that the space $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)$ and its dual space can be characterized as:

$$[\boldsymbol{H}_0^{r',\,p'}(\mathbf{curl},\,\Omega)]'=\{\boldsymbol{F}+\mathbf{curl}\,\boldsymbol{\psi},\ \ \boldsymbol{F}\in\boldsymbol{L}^r(\Omega),\ \ \boldsymbol{\psi}\in\boldsymbol{L}^p(\Omega)\}. \tag{5.45}$$

Theorem 5.7. *Let $\boldsymbol{f}$, $\boldsymbol{g}$ and π_0 such that:*

$$\boldsymbol{f}\in[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]',\quad \boldsymbol{g}\in\boldsymbol{W}^{1-1/p,p}(\Gamma),\quad \pi_0\in W^{1-1/p,p}(\Gamma),$$

with $r\le p$ and $\frac{1}{r}\le\frac{1}{p}+\frac{1}{3}$. Then, the problem $(\mathcal{S}'_N)$ has a unique solution $\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega)$, $\pi\in W^{1,r}(\Omega)$ and constants $c_1,\ldots,c_I$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|_{W^{1,r}(\Omega)}\le C\big(\|\boldsymbol{f}\|_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl},\,\Omega)]'}\|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}+\|\pi_0\|_{W^{1-1/p,p}(\Gamma)}\big),$$

and $c_1,\ldots,c_I$ are given by (5.44)*, where we replace the duality bracket on Ω by*

$\langle \cdot, \cdot \rangle_\Omega = \langle \cdot, \cdot \rangle_{[\boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)]' \times \boldsymbol{H}_0^{r',p'}(\mathbf{curl}, \Omega)}$.

Proof. Due to the characterization (5.45), we can write $\boldsymbol{f}$ as $\boldsymbol{f} = \boldsymbol{F} + \mathbf{curl}\,\boldsymbol{\psi}$, where $\boldsymbol{F} \in \boldsymbol{L}^r(\Omega)$ and $\boldsymbol{\psi} \in \boldsymbol{L}^p(\Omega)$. By Theorem 5.6, the following problem:

$$\begin{cases} -\Delta\, \boldsymbol{w} + \nabla\, q = \mathbf{curl}\,\boldsymbol{\psi} \;\text{ and }\; \operatorname{div} \boldsymbol{w} = 0 & \text{in}\,\Omega, \\ \boldsymbol{w} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} & \text{on}\,\Gamma, \\ q = \pi_0 \;\text{ on } \Gamma_0 \quad \text{and} \quad q = \pi_0 + d_i & \text{on}\,\Gamma_i, \;\; 1 \le i \le I, \\ \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I, \end{cases}$$

has a unique solution $(\boldsymbol{w},\, q,\, \boldsymbol{d}) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^I$, with $\boldsymbol{d} = (d_1, \dots, d_I)$ where

$$d_i = -\langle \pi_0,\, \nabla\, q_i^N \cdot \boldsymbol{n}\rangle_\Gamma, \quad 1 \le i \le I,$$

(note that for any $1 \le i \le I$, $\langle \mathbf{curl}\,\boldsymbol{\psi},\; \nabla\, q_i^N\rangle_\Omega = 0$). Again, by Theorem 5.6, the following problem

$$\begin{cases} -\Delta\, \boldsymbol{z} + \nabla\, \theta = \boldsymbol{F} \;\text{ and }\; \operatorname{div} \boldsymbol{z} = 0 & \text{in}\,\Omega, \\ \boldsymbol{z} \times \boldsymbol{n} = \boldsymbol{0} & \text{on}\,\Gamma, \\ \theta = 0 \;\text{ on } \Gamma_0 \quad \text{and} \quad \theta = e_i & \text{on}\,\Gamma_i, \;\; 1 \le i \le I, \\ \langle \boldsymbol{z} \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I, \end{cases}$$

has a unique solution $(\boldsymbol{z},\, \theta,\, \boldsymbol{e}) \in \boldsymbol{W}^{2,r}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$, where $e_i = \langle \boldsymbol{f},\, \nabla\, q_i^N\rangle_\Omega$. Observe that, since $\frac{1}{r} \le \frac{1}{p} + \frac{1}{3}$, $\boldsymbol{W}^{2,r}(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$. Then, $(\boldsymbol{u},\, \pi,\, \boldsymbol{c}) = (\boldsymbol{w} + \boldsymbol{z},\, q + \theta,\, \boldsymbol{d} + \boldsymbol{e})$ is the unique solution of the problem $(\mathcal{S}'_N)$. □

6. Helmholtz Decompositions

In this section, we assume that Ω is of class $\mathcal{C}^{1,1}$ and we give decompositions of vector fields $\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$. Our results may be regarded as an extension of the well-known De Rham-Hodge-Kodaira decomposition of $\mathcal{C}^\infty$-forms on compact Riemannian manifolds into $\boldsymbol{L}^p$-vector fields on Ω. We can find similar decompositions in Ref. 23, where the authors consider more regular domain with $\mathcal{C}^\infty$-boundary Γ. We can see also Ref. 27 for the case $p = 2$.

We introduce the space:

$$\boldsymbol{W}^{1,p}_\sigma(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\; \operatorname{div} \boldsymbol{v} = 0 \;\text{ in } \Omega\}$$

Theorem 6.1.

(i) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$*. Then, there exist* $\chi \in W^{1,p}(\Omega)$*,* $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_\sigma(\Omega) \cap \boldsymbol{X}^p_N(\Omega)$*,* $\boldsymbol{z} \in \boldsymbol{K}^p_T(\Omega)$ *such that* $\boldsymbol{u}$ *can be represented as:*

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\chi + \mathbf{curl}\,\boldsymbol{w}, \tag{6.1}$$

where $\boldsymbol{z}$ *is unique,* χ *is unique up to an additive constant and* $\boldsymbol{w}$ *is unique up to an additive element of* $\boldsymbol{K}^p_N(\Omega)$*. Moreover, we have the estimate:*

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)/\mathbb{R}} + \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}^p_N(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.2}$$

(ii) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$*. Then, there exist* $\chi \in W^{1,p}_0(\Omega)$*,* $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_\sigma(\Omega) \cap \boldsymbol{X}^p_T(\Omega)$*,* $\boldsymbol{z} \in \boldsymbol{K}^p_N(\Omega)$ *such that* $\boldsymbol{u}$ *can be represented as:*

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\chi + \mathbf{curl}\,\boldsymbol{w}, \tag{6.3}$$

where $\boldsymbol{z}$ *and* χ *are unique and* $\boldsymbol{w}$ *is unique up to an additive element of* $\boldsymbol{K}^p_T(\Omega)$*. Moreover, we have the estimate:*

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)} + \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}^p_T(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.4}$$

Proof.

(i) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. The scalar potential $\chi \in W^{1,p}(\Omega)$ is taken as a weak solution of the following problem:

$$\operatorname{div}(\nabla\chi - \boldsymbol{u}) = 0 \text{ in } \Omega, \quad (\nabla\chi - \boldsymbol{u})\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \tag{6.5}$$

or equivalently of

$$\forall \mu \in W^{1,p'}(\Omega), \quad \int_\Omega \nabla\chi\cdot\nabla\mu\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{u}\cdot\nabla\mu\,\mathrm{d}\boldsymbol{x}. \tag{6.6}$$

Such a scalar function χ as (6.6) is unique up to an additive constant and satisfies the estimate:

$$\|\chi\|_{W^{1,p}(\Omega)/\mathbb{R}} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.7}$$

Next, the vector potential $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_\sigma(\Omega) \cap \boldsymbol{X}^p_N(\Omega)$ can be derived from Proposition 5.1 and the point *ii)* of Remark 5.1. For $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, we take $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$-\Delta\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}, \quad \operatorname{div}\boldsymbol{w} = 0 \text{ in } \Omega \text{ and } \boldsymbol{w}\times\boldsymbol{n} = \mathbf{0} \text{ on } \Gamma.$$

The vector potential $\boldsymbol{w}$ is unique up to an additive element of $\boldsymbol{K}^p_N(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{w}\|_{W^{1,p}(\Omega)/\boldsymbol{K}^p_N(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{6.8}$$

Finally, let us define $\boldsymbol{z} = \boldsymbol{u} - \nabla \chi - \mathbf{curl}\, \boldsymbol{w}$. Then $\boldsymbol{z} \in \boldsymbol{K}^p_T(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} \leq \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla \chi\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\, \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}, \tag{6.9}$$

which yields the representation (6.1) of $\boldsymbol{u}$. The estimate (6.2) is a consequence of (6.7), (6.8) and (6.9).

(ii) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Compared with (6.5), the scalar potential $\chi \in W^{1,p}_0(\Omega)$ is taken as the weak solution of the Dirichlet problem:

$$\Delta \chi = \operatorname{div} \boldsymbol{u} \ \text{ in } \Omega, \quad \chi = 0 \ \text{ on } \Gamma.$$

Such a scalar function χ is unique and satisfies the estimate:

$$\|\chi\|_{W^{1,p}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

The vector potential $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_\sigma(\Omega) \cap \boldsymbol{X}^p_T(\Omega)$ can be derived from Proposition 4.2. For $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ we take $\boldsymbol{w} \in \boldsymbol{W}^{1,p}(\Omega)$ such that:

$$\begin{cases} \Delta \boldsymbol{w} = \mathbf{curl}\, \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega, \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, \quad (\mathbf{curl}\, \boldsymbol{w} - \boldsymbol{u}) \times \boldsymbol{n} = 0 & \text{on } \Gamma. \end{cases}$$

The vector potential $\boldsymbol{w}$ is unique up to an additive element of $\boldsymbol{K}^p_T(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{w}\|_{W^{1,p}(\Omega)/\boldsymbol{K}^p_T(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

Let us define $\boldsymbol{z} = \boldsymbol{u} - \nabla \chi - \mathbf{curl}\, \boldsymbol{w}$. Then, similarly to the proof of the above i) we obtain the representation (6.3) of $\boldsymbol{u}$ and the estimate (6.4). □

Remark 6.1.

(i) Note that in the representation (6.1), if Ω is simply connected then $\boldsymbol{z} = \boldsymbol{0}$, situation that can be the same in (6.3) if we suppose that the boundary Γ is connected or in other words without holes.

(ii) In the decomposition (6.1), $\boldsymbol{z}$ being a divergence-free vector field with a zero normal trace on the boundary. We know from Theorem 4.1 that $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{\psi}$ with $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p}(\Omega)$, $\operatorname{div} \boldsymbol{\psi} = 0$ in Ω and $\boldsymbol{\psi} \cdot \boldsymbol{n} = 0$ on Γ.

(iii) In the decomposition (6.3), $\boldsymbol{z}$ being an element of $\boldsymbol{K}^p_N(\Omega)$. We know then that $\boldsymbol{z}$ is a gradient of a function of $W^{1,p}(\Omega)$.

An immediate consequence of the above theorem is the following result.

Corollary 6.1.

By the unique decompositions (6.1) *and* (6.3)*, we have two kinds of direct sums:*

$$\boldsymbol{L}^p(\Omega) = \boldsymbol{K}^p_N(\Omega) \oplus \boldsymbol{H}_1 \oplus \boldsymbol{H}_3 \oplus \boldsymbol{H}^p_0(\mathrm{div},\,\Omega), \tag{6.10}$$

$$\boldsymbol{L}^p(\Omega) = \boldsymbol{K}^p_T(\Omega) \oplus \boldsymbol{H}_0 \oplus \boldsymbol{H}_1 \oplus \boldsymbol{H}_2, \tag{6.11}$$

where

$$\begin{aligned}
\boldsymbol{H}_0 &= \big\{\boldsymbol{v} \in \boldsymbol{H}^p_0(\mathrm{div},\,\Omega);\ \ \langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,\ \ \forall j = 1,\,\ldots,J\big\},\\
\boldsymbol{H}_1 &= \big\{\nabla\chi\,;\ \ \chi \in W^{1,p}_0(\Omega)\big\}, \quad \boldsymbol{H}_2 = \big\{\nabla q\,;\ \ q \in W^{1,p}(\Omega),\ \ \Delta q = 0\big\},\\
\boldsymbol{H}_3 &= \big\{\nabla\theta;\ \ \theta \in W^{1,p}(\Omega),\ \ \Delta\theta = 0,\ \ \langle\frac{\partial\theta}{\partial\boldsymbol{n}},\,1\rangle_{\Gamma_i} = 0,\ \ \forall i = 1,\ldots,I\big\}.
\end{aligned}$$

Proof.

(i) The direct sum (6.10) is a consequence of the representation formula (6.3) with the uniqueness. Indeed, suppose that $\boldsymbol{u}$ is decomposed as in (6.3) and let $\theta \in W^{1,p}(\Omega)$ solution of the following problem

$$\mathrm{div}\,(\nabla\theta - \mathbf{curl}\,\boldsymbol{w}) = 0 \ \ \text{in}\,\Omega, \qquad \frac{\partial\theta}{\partial\boldsymbol{n}} = \mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{n} \ \ \text{on}\,\Gamma,$$

which is unique up to an additive constant. The function $\boldsymbol{y} = \mathbf{curl}\,\boldsymbol{w} - \nabla\theta$ belongs to $\boldsymbol{H}^p_0(\mathrm{div},\,\Omega)$. Moreover, since $\langle\mathbf{curl}\,\boldsymbol{w}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0$, we have $\langle\frac{\partial\theta}{\partial\boldsymbol{n}},\,1\rangle_{\Gamma_i} = 0$ for any $i = 0,\ldots,I$. Then, we can write $\boldsymbol{u}$ as

$$\boldsymbol{u} = \boldsymbol{z} + \nabla\chi + \nabla\theta + \boldsymbol{y}, \tag{6.12}$$

where $\theta \in \boldsymbol{H}_3$ and $\boldsymbol{y} \in \boldsymbol{H}^p_0(\mathrm{div},\,\Omega)$. This completes the proof of (6.10).

(ii) Now, we give the proof of (6.11). Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. There exists a unique $\chi \in W^{1,p}(\Omega)$ such that $\Delta\chi = \mathrm{div}\,\boldsymbol{u}$ in Ω and $\chi = 0$ sur Γ. Since $(\boldsymbol{u} - \nabla\chi)\cdot\boldsymbol{n} \in W^{-1/p,p}(\Gamma)$, then the following problem $\Delta q = 0$ in Ω and $\dfrac{\partial q}{\partial\boldsymbol{n}} = (\boldsymbol{u} - \nabla\chi)\cdot\boldsymbol{n}$ on Γ has a unique solution $q \in W^{1,p}(\Omega)$. Note that $\langle(\boldsymbol{u} - \nabla\chi)\cdot\boldsymbol{n},\,1\rangle_\Gamma = \int_\Omega \mathrm{div}\,(\boldsymbol{u} - \nabla\chi)\,\mathrm{d}\boldsymbol{x} = 0$. We set, $\boldsymbol{u}_1 = \nabla\chi$, $\boldsymbol{u}_2 = \nabla q$ and $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{u}_1 - \boldsymbol{u}_2$. Observe that $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{u}_1 + \boldsymbol{u}_2 \in \boldsymbol{H}^p_0(\mathrm{div},\,\Omega) \oplus \boldsymbol{H}_1 \oplus \boldsymbol{H}_2$. Now, it remains to prove that $\boldsymbol{H}^p_0(\mathrm{div},\,\Omega) = \boldsymbol{K}^p_T(\Omega) \oplus \boldsymbol{H}_0$. For this, it suffices to observe that any function $\boldsymbol{v} \in \boldsymbol{H}^p_0(\mathrm{div},\,\Omega)$ can be written as

$$\boldsymbol{v} = \boldsymbol{w} + \sum_{j=1}^{J}\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q^T_j,$$

where $\boldsymbol{w} = \boldsymbol{v} - \sum_{j=1}^{J}\langle\boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}\widetilde{\mathbf{grad}}\,q^T_j \in \boldsymbol{H}_0$. This completes the proof. □

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